Solution of the vacuum Einstein equations in Synthetic Differential Geometry of Kock-Lawvere

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Auguest 30, 1999

ABSTRACT

The topos theory is a theory which is used for deciding a number of problems of theory of relativity, gravitation and quantum physics. It is known that topos-theoretic geometry can be successfully developed within the framework of Synthetic Differential Geometry of Kock-Lawvere (SDG), the models of which are serving the toposes, i.e. categories possessing many characteristics of traditional Theory of Sets. In the article by using ideas SDG, non-classical spherically symmetric solution of the vacuum Einstein equations is given.

Theoretical physics always tended operatively to use new ideas coming up from the mathematics. So it is not wonderful that new topos-theoretic mathematics [1, 2] was immediately called to deciding a number of problems of theory of relativity and gravitation [3, 4, 5, 6, 7] and quantum physics [8]. Formally, for instance, it is suitable to develop the topos-theoretic geometry within the framework of Synthetic Differential Geometry of Kock-Lawvere [2] (further for brevity we write SDG), models of which are serving toposes, i.e. categories possessing many characteristics of traditional Theory of Sets. Last theory was the basis of mathematics of the XX century. In the article by using ideas SDG, non-classical solution of the vacuum Einstein equations is given.

1 Intuitionistic theory of gravitation

Synthetic Differential Geometry of Kock-Lawvere [2] is built on the base of change the field of real numbers $\mathbb{R}$ on commutative ring $\mathbf{R}$, allowing to define on him differentiation, integrating and "natural numbers". It is assumed that there exists D such that $D = \{x \in \mathbf{R} \mid x^2 = 0\}$ and that following the Kock-Lawvere axiom is held:

for any $g : D \rightarrow \mathbf{R}$ it exist the only $a, b \in \mathbf{R}$ such that for any $d \in D$ $g(d) = a + d \cdot b$.

This means that any function in given geometry is differentiable, but "the law of excluded middle" is false. In other words, intuitionistic logic acts in SDG. But on this way one is possible building an intuitionistic theory of gravitation in analogy with the General theory of Relativity of Einstein [5, 6, 7]. The elements of $d \in D$ are called infinitesimals, i.e. infinitesimal numbers. On the ring $\mathbf{R}$ we can look as on the field of real numbers $\mathbb{R}$ complemented by infinitesimals.

The vacuum Einstein equations in SDG in space-time $\mathbf{R}^4$ can be written with *nonzero* tensor of the energy. For instance,

$$R_{ik} - \frac{1}{2}g_{ik}(R - 2\Lambda) = \frac{8\pi G}{c^2} \, du_i u_k, \tag{1}$$

where density of matter $d \in D$ is arbitrarily taken infinitesimal [9]. For infinitesimals are holding relations which are impossible from standpoints of classical logic: $d \neq 0$, $d \leq 0$ & $d \geq 0$ and $-\epsilon < d < \epsilon$ for any $\epsilon \in \mathbf{R}, \epsilon > 0$. Such non-classical density of vacuum matter will consistent with zero in right part of the Einstein's equations in the case of the vacuum in classical General

theory of Relativity. For this one is sufficiently to consider SDG in so named well-adapted models, in which we can act within the framework of classical logic. For instance, in smooth topos $\mathbf{Set^{L^{op}}}$, where $\mathbf{L}$ category C^∞-rings [10], the equations (1) at stage of locus $\ell A = \ell(C^\infty(\mathbb{R}^n)/I)$, I is a certain ideal of C^∞-smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$, have the form

$$R_{ik}(a) - \frac{1}{2}g_{ik}(a)(R(u) - 2\Lambda(a)) = \frac{8\pi G}{c^2}\ d(a)u_i(a)u_k(a)\ mod\ I, \qquad (2)$$

where $a \in \mathbb{R}^n$ in parenthesises shows that we have functions, but at stage $\mathbf{1} = \ell(C^\infty(\mathbb{R})/\{a\})$, equations (2) take a classical form with null (on $mod\ \{a\}$) tensor of the energy.

Note that an event x of the space-time $\mathbf{R}^4$ at stage ℓA is the class of C^∞-smooth vector functions $(X^0(a), X^1(a), X^2(a), X^3(a)) : \mathbb{R}^n \to \mathbb{R}^4$, where each function $X^i(a)$ is taken by $mod\ I$. The argument $a \in \mathbb{R}^n$ is some "hidden" parameter corresponding to the stage ℓA. Hence it follows that at stage of real numbers $\mathbf{R} = \ell C^\infty(\mathbb{R})$ of the topos under consideration an event x is described by just a C^∞-smooth vector function $(X^0(a), X^1(a), X^2(a), X^3(a)), a \in \mathbb{R}$. At stage of $\mathbf{R}^2 = \ell C^\infty(\mathbb{R}^2)$ an event x is 2-dimensional surface, i.e. a *string*. The classical four numbers $(x^0,\ x^1,\ x^2,\ x^3)$, the coordinates of the event x, are obtained at the stage $\mathbf{1} = \ell C^\infty(\mathbb{R}^0) = \ell C^\infty(\mathbb{R})/\{a\}$ (the ideal $\{a\}$ allows one to identify functions if their values at 0 coincide), i.e., $x^i = X^i(0), i = 0, 1, 2, 3$.

2 Spherically symmetrical field

We have the Einstein equations describing the gravitational field created by certain material system

$$R_{ik} - \frac{1}{2}g_{ik}(R - 2\Lambda) = \kappa T_{ik}$$

Here $R_{ik} = R^l_{ilk}$, $R = g^{ik}R_{ik}$, $\kappa = 8\pi G/c^4$.

Consider case, when gravitational field possesses a central symmetry. Central symmetry of field means that interval of space-time can be taken in the form

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2)$$

Note that such type of metric does not define else choice of time coordinate by unambiguous image: given metric can be else subject to any transformation of type $t = f(t')$ not containing r.

All calculations are conducted also as in the classical case. Herewith we consider components of metric tensor by invertible values in **R**. For the Christoffel coefficients we have the usual formula

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

Hence we have

$$\Gamma^1_{11} = \frac{\lambda'}{2}, \Gamma^0_{10} = \frac{\nu'}{2}, \Gamma^2_{33} = -\sin\theta\cos\theta, \Gamma^0_{11} = \frac{\dot\lambda}{2}e^{\lambda-\nu}, \Gamma^1_{22} = -re^{-\lambda}, \Gamma^1_{00} = \frac{\nu'}{2}e^{\nu-\lambda},$$

$$\Gamma^2_{12} = \Gamma^3_{13} = 1/r, \Gamma^3_{33} = \text{ctg}\theta, \Gamma^0_{00} = \frac{\dot\nu}{2}, \Gamma^1_{10} = \frac{\dot\lambda}{2}, \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}.$$

Here the prime means differentiation with respect to r, and dot means differentiation with respect to t.

Tensor of Ricci is also calculated with help of known formula

$$R_{ik} = \frac{\partial \Gamma^l_{ki}}{\partial x_l} - \frac{\partial \Gamma^l_{il}}{\partial x_k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

The Einstein's equations have the form:

$$-e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda = \kappa T^1_1 \tag{3}$$

$$-\frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu'-\lambda'}{r} - \frac{\nu'\lambda'}{2}\right) + \frac{1}{2}e^{-\nu}\left(\ddot\lambda + \frac{\dot\lambda^2}{2} - \frac{\dot\nu\dot\lambda}{2}\right) - \Lambda = \kappa T^2_2 = \kappa T^3_3 \tag{4}$$

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} - \Lambda = \kappa T^0_0 \tag{5}$$

$$-e^{-\lambda}\frac{\dot\lambda}{r} = \kappa T^1_0 \tag{6}$$

Equation (4), as it is well known [11], is corollary of equations (3), (5), (6) and the law of conservation

$$T^{ik}_{;k} = 0. \tag{7}$$

So further the equation (4) will be omited.

2.1. Field of vacuum. Consider now important example of gravitational field in the vacuum. For this we will take $T^i_k = c^2\rho u^i u_k$, i.e. tensor of the energy

3

of dust matter. Here ρ is density of dust in the space which will consider further constant value. Suppose that dust is described in coordinate system in which $u_i = (\text{e}^{-\frac{\nu}{2}}, 0, 0, 0)$, $u^k = g^{ik}u_i = (\text{e}^{\frac{\nu}{2}}, 0, 0, 0)$. So $T_0^0 = c^2\rho$, $T_1^1 = T_2^2 = T_3^3 = 0$ and equations (3),(5),(6) will take following forms

$$-\text{e}^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda = 0 \tag{8}$$

$$-\text{e}^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} - \Lambda = \kappa c^2\rho \tag{9}$$

$$-\text{e}^{-\lambda}\frac{\dot{\lambda}}{r} = 0 \tag{10}$$

By using form of tensor T_k^i and equation (10) we rewrite the equation (7) as follows

$$\rho\nu' = 0 \tag{11}$$

Try to solve equations (8)-(10) using equation (11). From equation (10) it follows that $\lambda(r, t) = \lambda(r)$, i.e. λ does not depend on coordinate t. As far as ρ and Λ are constants, the equation (9) can be easy integrated. Really, by taking $\text{e}^{-\lambda}$ for u, we get

$$u'r + u = 1 - (\Lambda + \kappa c^2\rho)r^2 \tag{12}$$

Solution of uniform equation $u'r + u$ has form $u = \dfrac{A}{r}$, where $A = const.$ Thereby, for non-uniform equation we will get $u = \dfrac{A(r)}{r}$. Substituting this in (12), we have for $A(r)$:

$$A'(r) = 1 - (\Lambda + \kappa c^2\rho)r^2$$

Solution of this equation is function

$$A(r) = r - \frac{(\Lambda + \kappa c^2\rho)r^3}{3} + C.$$

Thence

$$u(r) = 1 - \frac{(\Lambda + \kappa c^2\rho)r^2}{3} + \frac{C}{r}$$

or

$$\text{e}^{-\lambda} = 1 - \frac{(\Lambda + \kappa c^2\rho)r^2}{3} + \frac{C}{r} \tag{13}$$

4

Here C is a constant of integrating.

Hereinafter, to find an expression for ν, we need integrate an equation (8). But for the beginning we consider an equation (11).

Notice that $\rho = d$, $\nu = d$ under any $d \in D$ are its solutions. Thereby, from existence of such objects as D, D_2, $D(2)$, $\triangle$ and etc [2], it follows that except classical its solutions

$$(\rho = 0 \ \& \ \nu \neq 0) \vee (\nu = 0 \ \& \ \rho \neq 0) \vee (\rho = 0 \ \& \ \nu = 0)$$

there exist and the others, non-classical one. For example, ρ and ν that are inseparable from the zero (x is separable from the zero, if there exists a natural number n such that $(1/n) < x \ \vee \ x < -(1/n)$). The First from classical cases above gives well-known the classical Schwarzschild solution.

Consider non-classical case of deciding an equation (11), when both values ρ and ν are simultaneously inseparable from the zero. Substituting (13) in (8) and considering (11) we get

$$\frac{\nu'}{r}(1 + \frac{C}{r} - \frac{\Lambda r^2}{3}) + \frac{2}{3}\Lambda - \frac{\kappa c^2 \rho}{3} + \frac{C}{r^3} = 0 \tag{14}$$

Thence easy notice that $\frac{2}{3}\Lambda + \frac{C}{r^3}$ is inseparable from the zero. Besides, when considering this expressions in topos $\mathbf{Set}^{\mathbf{L_{op}}}$ at stage $\mathbf{1}$ this expression becomes an equal to zero that is possible in that case only, when and Λ and C at this stage are zero. Thereby, we conclude that C and Λ are also inconvertible, but hence and $\frac{C}{r} - \frac{\Lambda r^2}{3}$ is inconvertible. By using now (11), we will transform (14) to the form

$$\nu'(1 + \frac{C}{r} - \frac{\Lambda r^2}{3} + \frac{\kappa c^2 \rho r^2}{6}) = \frac{\kappa c^2 \rho r}{3} - \frac{2}{3}\Lambda \cdot r - \frac{C}{r^2}$$

or, that is equivalent,

$$\nu' = \frac{\dfrac{\kappa c^2 \rho r}{3} - \dfrac{2}{3}\Lambda \cdot r - \dfrac{C}{r^2}}{1 + \dfrac{C}{r} - \dfrac{\Lambda r^2}{3} + \dfrac{\kappa c^2 \rho r^2}{6}}, \tag{15}$$

Deciding equation (15) we find that

$$\nu = \ln \left| 1 + \frac{C}{r} - \frac{\Lambda r^2}{3} + \frac{\kappa c^2 \rho r^2}{6} \right| + f(t),$$

where $f(t)$ is arbitrary function that is depending only on coordinate t. On the strength of that that we left for itself else possibility of arbitrary transformation of time $t = g(t')$, which is equivalent addition to ν an arbitrary functions of time, $f(t)$ can always be made to be equal to zero. Hence, not limiting generalities, it is possible to consider that

$$\nu = \ln \left| 1 + \frac{C}{r} - \frac{\Lambda r^2}{3} + \frac{\kappa c^2 \rho r^2}{6} \right| \tag{16}$$

Substituting these values for λ and ν in expression for ds^2, we get that

$$ds^2 = \left(1 + \frac{(\kappa c^2 \rho - 2\Lambda)r^2}{6} + \frac{C}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{(\Lambda + \kappa c^2 \rho)r^2}{3} + \frac{C}{r}} -$$

$$-r^2(d\theta^2 + \sin^2 \theta \cdot d\varphi^2) \tag{17}$$

This metric can be called the non-classical Schwarzschild solution of the Einstein equations

Suppose that gravitational field has no singularity in all space. This means that metric has no singularity in $r = 0$. So we shall consider that C is zero. Coming from this and multiplying the right and left parts of equations (14) on ρ, we get that $2\Lambda\rho = \kappa c^2 \rho^2$ and, besides, Λ is inconvertible value of ring $\mathbf{R}$.

In other words, matter has non-classical density, and its gravitational field has the form

$$ds^2 = \left(1 + \frac{(\kappa c^2 \rho - 2\Lambda)r^2}{6} \right) dt^2 - \frac{dr^2}{1 - \frac{(\Lambda + \kappa c^2 \rho)r^2}{3}} - r^2(d\theta^2 + \sin^2 \theta \cdot d\varphi^2)$$

In topos $\mathbf{Set^{L^{op}}}$ at stage $\mathbf{1}$ this metric complies with the metric of the Minkowski space-time. Roughly speaking, non-classical "dust" vacuum has the "infinitesimal" weak gravitational field.

2.2. Field of gas ball. Suppose that gravitational field was created by spherical gas ball of radius a with tensor of the energy $\tilde{T}_{ik}$. From formula (5) under condition of absence a singularity of matter of the form $\lambda|_{r=0} = 0$ we get

$$\lambda = -\ln \left(1 - \frac{\kappa}{r} \int_0^r \tilde{T}_0^0 r^2 dr - \frac{\Lambda r^2}{3} \right)$$

Outside of the ball we have the vacuum with $\tilde{T}_{ik} = c^2 \rho u_i u_k$ and with gravitational field that was studied in the preceding point. So it is possible to use

expression (13), from which it follows that

$$\lambda = -\ln\left(1 - \frac{\Lambda + \kappa c^2 \rho}{3}r^2 + \frac{C}{r}\right)$$

Comparing both expressions under $r = a$, we find that

$$C = \kappa \cdot \left(\frac{c^2 \rho a^3}{3} - \int\limits_0^a \tilde{T}_0^0 r^2 dr\right) \qquad (18)$$

By using that C and ρ are inconvertible and (18), we get that $\int_0^a \tilde{T}_0^0 r^2 dr$ is inconvertible. This is possible only in two cases: 1) $\tilde{T}_0^0$ is inseparable from the zero; 2) a is inseparable from the zero.

Thereby, the following theorem is true.

Theorem. *Let gas ball possesses classical nonzero density $(\tilde{T}_0^0 \neq 0)$ and creates external spherically symmetrical gravitational field (17) with dust infinitesimal density ρ. Then ball has infinitesimal sizes.*

It is interesting that in the classical case the Schwarzschild solution was found in the suggestion that gravitational field is created by the ball that is so naming material point, which is not having sizes. Such situation was characterized by the word "simplification". In non-classical case a material point gets wholly legal sizes, but they will be described infinitesimal numbers.

Notice that unlike classical solution, constant C can not so simply be expressed through the mass of ball. Really, following classical procedure, we are noting that on greater distances, where field is weak, the field must be described by the Newton's Law. Hence, $g_{00} = 1 - \dfrac{2Gm}{c^2 r}$, where m is a mass of ball. On the other hand, $g_{00} = 1 + \dfrac{\kappa c^2 \rho - 2\Lambda}{6}r^2 + \dfrac{C}{r}$. Thence it is seen that $C = \dfrac{2\Lambda - \kappa c^2 \rho}{6}r^3 - \dfrac{2Gm}{c^2}$. This gives contradiction with $C = const$.

In topos $\mathbf{Set}^{\mathbf{L^{op}}}$ at stage $\mathbf{1}$ metric (17) complies with the metric of the Partial theory of Relativity. So cosmological model with this metric can be called a generalized model of the Partial theory of Relativity .

References

[1] Goldblatt, R. Topoi. The categorical analisys of logic. Amsterdam: North-Holland Publ. Comp., 1979.

[2] Kock A. Synthetic Differential Geometry. Cambridge University Press, 1981.

[3] Guts, A.K. A topos-theoretic approach to the foundations of Relativity theory // Soviet Math. Dokl.- 1991.-V.43, No.3.-P.904-907.

[4] Trifonov, V. Linear Solution of the Four-Dimensionality Problem // Europhys. Lett. 1995. V.32, N.8, P.621-626.

[5] Guts, A.K. Intuitionistic Theory of space-time // International geometric school-seminar of memory N.V.Efimov. Avstracts. – Rostov-on Don, 1996. P.87-88.

[6] Grinkevich, E.B.. Synthetic Differential Geometry: A Way to Intuitionistic Models of General Relativity in Toposes – Paper gr-qc/9608013

[7] Guts, A.K., Grinkevich E.B. Toposes in General Theory of Relativity. – Paper gr-qc/9610073.

[8] Isham, C.J. Topos Theory and Consistent Histories: The Internal Logic of the Set of all –Paper gr-qc/9607069

[9] Guts, A.K. Vacuum equations of Einstein in Synthetic Differential Geometry of Kock-Lawvere. Abstracts of X Russian gravitational conferences, Vladimir, 1999.

[10] Moerdijk, I., Reyes, G.E. Models for Smooth Infinitesimal Analysis. Springer-Verlag, 1991.

[11] Synge, J.L. Relativity: the General theory. Amsterdam: North-Holland Publ. Comp., 1960.