Interpretation of intuitionistic solution of the vacuum Einstein equations in smooth topos

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ABSTRACT

The topos theory is a theory which is used for deciding a number of problems of theory of relativity, gravitation and quantum physics. In the article spherically symmetric solution of the vacuum Einstein equations in the Intuitionistic theory of Gravitation at different stages of smooth topos $\mathbf{Set}^{\mathbf{L}^{op}}$ is considered. Infinitesimal "weak" gravitational field can be strong at some stagies, for which we have the additional dimensions. For example, the cosmological constant is not constant with respect to additional dimensions. Signature of space-time metric can depend of density of vacuum and cosmological constant.

This article applies the Topos theory [1] to theory of space-time. Other applications can be found in [3, 5, 4, 6, 7, 8, 9].

1 Intuitionistic theory of gravitation

Intuitionistic theory of gravitation is based on Synthetic Differential Geometry of Kock-Lawvere (SDG) [2]. SDG is built on the base of change the field of real numbers $\mathbb{R}$ on commutative ring $\mathbf{R}$, allowing to define on him differentiation, integrating and "natural numbers". It is assumed that there exists D such that $D = \{x \in \mathbf{R} \mid x^2 = 0\}$ and that following the Kock-Lawvere axiom is held:

for any $g : D \to \mathbf{R}$ *it exist the only* $a, b \in \mathbf{R}$ *such that* $g(d) = a + d \cdot b$ *for any* $d \in D$.

This means that any function in given geometry is differentiable, but "the law of excluded middle" is false. In other words, intuitionistic logic acts in SDG. But on this way one is possible building an intuitionistic theory of gravitation in analogy with the General theory of Relativity of Einstein [5, 6, 7]. The elements of $d \in D$ are called infinitesimals, i.e. infinitesimal numbers. On the ring $\mathbf{R}$ we can look as on the field of real numbers $\mathbb{R}$ complemented by infinitesimals.

The vacuum Einstein equations in SDG in space-time $\mathbf{R}^4$ can be written with *nonzero* tensor of the energy. For instance,

$$R_{ik} - \frac{1}{2} g_{ik}(R - 2\Lambda) = \frac{8\pi G}{c^2}\, du_i u_k, \tag{1}$$

where density of matter $d \in D$ is arbitrarily taken infinitesimal [10]. For infinitesimals are holding relations which are impossible from standpoints of classical logic: $d \neq 0$,& $d \leq 0$ and formulas $d = 0$, $d \neq 0$ are not valied. Such non-classical density of vacuum matter will consistent with zero in right part of the Einstein's equations in the case of the vacuum in classical General theory of Relativity. For this one is sufficiently to consider SDG in so named well-adapted models, in which we can act within the framework of classical logic. For instance, in smooth topos $\mathbf{Set}^{\mathbf{L^{op}}}$, where $\mathbf{L}$ is category of *loci*, i.e. is the opposite category of category of finitely generated C^∞-rings [11], the equations (1) at stage of locus $\ell A = \ell(C^\infty(\mathbb{R}^n)/I)$, I is a certain ideal of C^∞-smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$, have the form

$$R_{ik}(a) - \frac{1}{2} g_{ik}(a)(R(u) - 2\Lambda(a)) = \frac{8\pi G}{c^2}\, d(a) u_i(a) u_k(a) \bmod I, \tag{2}$$

where $a \in \mathbb{R}^n$ in parenthesises shows that we have functions, but at stage $\mathbf{1} = \ell(C^\infty(\mathbb{R})/\{a\})$, equations (2) take a classical form with null (on mod $\{a\}$) tensor of the energy.

2 Spherically symmetrical vacuum field

We have the Einstein equations describing the gravitational field created by certain material system

$$R_{ik} - \frac{1}{2}g_{ik}(R - 2\Lambda) = \kappa c^2 \rho u_i u_k, \tag{3}$$

Here $\kappa = 8\pi G/c^4$, ρ is density of dust in the space which will consider further constant value. Suppose that dust is described in coordinate system in which $u_i = (\mathrm{e}^{-\frac{\nu}{2}}, 0, 0, 0)$, $u^k = g^{ik}u_i = (\mathrm{e}^{\frac{\nu}{2}}, 0, 0, 0)$. Consider case, when gravitational field possesses a central symmetry. Central symmetry of field means that interval of space-time can be taken in the form

$$ds^2 = \mathrm{e}^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2)$$

In our paper [12] we found the following vacuum solution of the equations (3) for which

$$\rho\nu' = 0. \tag{1}$$

1) Well-known the classical Schwarzschild solution when $\rho = 0$ and $\nu' \neq 0$

$$ds^2 = \left(1 - \frac{\Lambda r^2}{3} + \frac{C}{r}\right)dt^2 - \frac{dr^2}{1 - \dfrac{\Lambda r^2}{3} + \dfrac{C}{r}} -$$

$$-r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2). \tag{4}$$

2) Non-classical case when both values ρ and ν' are simultaneously inseparable from the zero, i.e. are infinitesimals

$$ds^2 = \left(1 + \frac{(\kappa c^2\rho - 2\Lambda)r^2}{6} + \frac{C}{r}\right)dt^2 - \frac{dr^2}{1 - \dfrac{(\Lambda + \kappa c^2\rho)r^2}{3} + \dfrac{C}{r}} -$$

$$-r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2) \tag{5}$$

This metric can be called the non-classical Schwarzschild solution of the Einstein equations. But here Λ and C are infinitesimals. So (5) is *infinitesimal* gravitational field, which is not weak in some sence (see below §5).

3) Suppose that gravitational field has no singularity in all space. This means that metric has no singularity in $r = 0$. So we shall consider that C is zero. But it is proved that

$$2\Lambda\rho = \kappa c^2\rho^2 \tag{6}$$

and, besides, Λ is inconvertible value of ring **R**.

In other words, matter has non-classical density, and its gravitational field has the form

$$ds^2 = \left(1 + \frac{(\kappa c^2\rho - 2\Lambda)r^2}{6}\right)dt^2 - \frac{dr^2}{1 - \dfrac{(\Lambda + \kappa c^2\rho)r^2}{3}} - r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2) \qquad (7)$$

Below we consider solutions (5), (7) in models which are different toposes, or, more exactly in categories of presheaves $\mathbf{Set}^{\mathcal{C}^{op}}$, where $\mathcal{C}$ is such subcategory of category $\mathbf{L}$ [11] that ring $\mathbf{R} = \mathrm{Hom}_{\mathbf{L}}(-, \ell C^\infty(\mathbb{R}))$ is local and Archimedean [1].

3 Stagies

Synthetic Differential Geometry uses "naive" style, i.e. contains term "element", or the set-theoretic formulas of the form $a \in A$. So it is necessary to be able to understand this naive writing as referering to cartesian closed categories, and to toposes in particular, because the Kock-Lawvere axiom has no models in the category of sets. The method for deciding this problem is introducing of *generalized element* $b \in_X B$, that is, a map $X \to B$, where X is an arbitrary object, called the *stages of definition*, or the *domain of variation* of the element b. The "classical" element is a map $\mathbf{1} \to B$. As A.Kock writes "when thinking in terms of physics (of which geometry of space forms a special case), reason for the name "domain of variation" (instead of "stage of definition") becomes clear: for a non-atomistic point of view, a body B is not described just in terms of its "atoms" $b \in B$, that is $\mathbf{1} \to B$, but in terms of "particles" of varying size X, or in terms of motions that take place in B and are parametrized by a temporal extent X; both of these situations being described by maps $X \to B$ for suitable domain of variation X [2]. In our case the role of "body" B will play a gravitational vacuum field g_{ik}, or geometry of space-time, which we will study at different stages, or with respect to different points of view concerning the possible geometric structure of the World.

In the case of topos $\mathbf{Set}^{\mathbf{L}}$ the concept of stage is realized with the help of the folowing method.

There exists the Yoneda embeding [13, p.26]

$$y : \mathbf{L} \to \mathbf{Set}^{\mathbf{L}^{op}},$$

$$y(\ell A)(\ell B) = \mathrm{Hom}_{\mathbf{L}}(\ell B, \ell A)$$

and for a morphism $\alpha : \ell B \to \ell C$

$$y(\ell A)(\alpha) : \mathrm{Hom}_{\mathbf{L}}(\ell C, \ell A) \to \mathrm{Hom}_{\mathbf{L}}(\ell B, \ell A),$$

[1] for example, categories of closed and germ-detrmined ideals are such subcategories [11]

$$y(\ell A)(\alpha)(u) = u \circ \alpha, u : \ell C \to \ell A.$$

Briefly for the Yoneda embedings we write

$$y(\ell A) = \mathrm{Hom}_{\mathbf{L}}(-, \ell A).$$

Instead of $y(\ell A)$ we shall write simply ℓA. So if ring $\mathbf{R}$ is $\ell C^\infty(\mathbb{R})$ then

$$\mathbf{R} \equiv y(\mathbf{R}) = \mathrm{Hom}_{\mathbf{L}}(-, \ell C^\infty(\mathbb{R})).$$

Hence element of ring $\mathbf{R}$, i.e. intuitionistic *real number* can be represented by arbitrary morphism of the form $\ell A \to \ell C^\infty(\mathbb{R})$. We say in such case that one have *real at stage* ℓA. It means that metric (5) must be considered at different stages. For example, at stage $\ell A = \ell C^\infty(\mathbb{R}^n)/I$, where I is some ideal of ring $C^\infty(\mathbb{R}^n)$.

Note that an event x of the space-time $\mathbf{R}^4$ at stage ℓA is the class of C^∞-smooth vector functions $(X^0(a), X^1(a), X^2(a), X^3(a)) : \mathbb{R}^n \to \mathbb{R}^4$, where each function $X^i(a)$ is taken by mod I. The argument $a \in \mathbb{R}^n$ is some "hidden" parameter corresponding to the stage ℓA. Hence it follows that at stage of real numbers $\mathbf{R} = \ell C^\infty(\mathbb{R})$ of the topos under consideration an event x is described by just a C^∞-smooth vector function $(X^0(a), X^1(a), X^2(a), X^3(a)), a \in \mathbb{R}$. At stage of $\mathbf{R}^2 = \ell C^\infty(\mathbb{R}^2)$ an event x is 2-dimensional surface, i.e. a *string*. The classical four numbers $(x^0,\ x^1,\ x^2,\ x^3)$, the coordinates of the event x, are obtained at the stage $\mathbf{1} = \ell C^\infty(\mathbb{R}^0) = \ell C^\infty(\mathbb{R})/\{a\}$ (the ideal $\{a\}$ allows one to identify functions if their values at 0 coincide), i.e., $x^i = X^i(0), i = 0, 1, 2, 3$.

There exists a number of types of infinitesimals: first-order infinitesimals $D = \{x \in \mathbf{R} | x^2 = 0\}$, k^{th}-order infinitesimals $D_k = \{x \in \mathbf{R} | x^{k+1} = 0\}$, and the infinitesimals $\Delta = \{x \in \mathbf{R} | f(x) = 0, all f \in m^g_{\{0\}}\}$, where $m^g_{\{0\}}$ is the ideal of functions having zero germ at 0, i.e. vanishing in a neighborhood of 0,

$$D \subset D_2 \subset \ldots \subset D_k \subset \ldots \subset \Delta.$$

Infinitesimals $\rho, \Lambda \in \Delta$ at stage $\ell A = \ell C^\infty(\mathbb{R}^n)/I$ are the classes $\rho(a)$ mod I, $\Lambda(a)$ mod I such that for every $\phi \in m^g_{\{0\}},\ \ \phi \circ \rho \in I,\ \ \phi \circ \Lambda \in I$, where $\rho : \mathbb{R}^n \to \mathbb{R}$ [11, p.77], or

$$(\phi \circ \rho)(a) = 0 \text{ mod } I, \qquad (\phi \circ \Lambda)(a) = 0 \text{ mod } I$$

The condition (6) has the form

$$2\Lambda(a)\rho(a) - \kappa c^2 \rho^2(a) = 0 \text{ mod } I. \tag{6$'$}$$

Let us to consider the forms of metrics (5), (7) at different stages.

3.1 Stage 1

For classical General Theory of Relativity we have stage $\mathbf{1} = \ell C^\infty(\mathbb{R}^0) = \ell C^\infty(\mathbb{R})/\{a\}$. At this stage metric (7) is metric of Minkowski space-time

$$g_{ik}(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix},$$

i.e. Λ and ρ are equal to zero. In fact,

$$\phi(\rho(a)) = \phi(\rho(0)) + \phi'(\rho(0))\rho'(0)a + o(|a|). \tag{8}$$

Since $\phi(\rho(a)) \in I = \{a\}$ then $\phi(\rho(0)) = 0$. Hence $\rho(0) = 0$, because $\phi \in m^g_{\{0\}}$. Then $\rho \bmod I = 0$. Similaly $\Lambda \bmod I = 0$, $C \bmod I = 0$

3.2 Stage $D = \ell C^\infty(\mathbb{R})/\{a^2\}$

In this case

$$g_{ik}(a) =$$

$$\begin{pmatrix} 1 + \dfrac{(\kappa c^2\rho_1 - 2\Lambda_1)}{6}\cdot ar^2 + \dfrac{C_1 a}{r} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{1 - \dfrac{(\kappa c^2\rho_1 + \Lambda_1)}{3}\cdot ar^2 + \dfrac{C_1 a}{r}} & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix}$$

In fact, it follows from (8) that $\phi(\rho(0)) = \phi'(\rho(0))\rho'(0) = 0$. Since $\phi|_U \equiv 0$, $\phi'|_U \equiv 0$ for some neighborhood of 0, then $\rho(0) = 0$. So $\rho \bmod I = \rho_1 a$, $\rho_1 \in \mathbb{R}$. Similaly $\Lambda \bmod I = \Lambda_1 a$, $\Lambda_1 \in \mathbb{R}$.

3.3 Stage $D_p = \ell C^\infty(\mathbb{R})/\{a^{p+1}\}$

Here

$$g_{00}(a) = 1 + \sum_{k=1}^{p}\left[\frac{(\kappa c^2\rho_k - 2\Lambda_k)}{6}\cdot r^2 + \frac{C_k}{r}\right]a^k$$

$$g_{11}(a) = -\frac{1}{1 - \sum\limits_{k=1}^{p}\left[\dfrac{(\kappa c^2\rho_k + \Lambda_k)}{3}\cdot r^2 - \dfrac{C_k}{r}\right]a^k}$$

and others g_{ik} are classical.

3.4 Stage $D_n(k) = \ell J_n^k = \ell C_0^\infty(\mathbb{R}^n)/m^{k+1}$

Let $m = f|f(0) = 0$ is maximal ideal of ring $\ell C_0^\infty(\mathbb{R}^n)$. Then

$$g_{00}(a,t,r,\varphi,\theta) = 1 + \sum_{l=1}^{k} \sum_{i_1,\dots,i_l=1}^{n} \left[\frac{(\kappa c^2 \rho_{i_1,\dots,i_l} - 2\Lambda_{i_1,\dots,i_l})}{6} r^2 + \frac{C_{i_1,\dots,i_l}}{r} \right] a_{i_1} \dots a_{i_l}$$

$$g_{11}(a,t,r,\varphi,\theta) = -\frac{1}{1 - \sum\limits_{l=1}^{k} \sum\limits_{i_1,\dots,i_l=1}^{n} \left[\frac{(\kappa c^2 \rho_{i_1,\dots,i_l} + \Lambda_{i_1,\dots,i_l})}{3} r^2 - \frac{C_{i_1,\dots,i_l}}{r} \right] a_{i_1} \dots a_{i_l}}$$

Here $a = (a_1, \dots, a_n)$.

3.5 Stage $\ell C^\infty(\mathbb{R}^2)/\{a_1 - a_2\}$

In this case functions Λ, ρ, C depend of one variable, for example, a_2, and vanishing at 0. Then

$$g_{00} = 1 + \frac{\kappa c^2 \rho(a_2) - 2\Lambda(a_2)}{6} r^2 + \frac{C(a_2)}{r}$$

$$g_{11} = -\frac{1}{1 - \frac{\kappa c^2 \rho(a_2) + \Lambda(a_2)}{3} r^2 + \frac{C(a_2)}{r}}$$

3.6 Stage $\ell C^\infty(\mathbb{R})/\{\sin \pi a, \cos \pi a\}$

If $\rho \in D$ then

$$\rho(a) = \frac{\alpha_0}{2} + \sum_{k=0}^{\infty} \alpha_k \cos k\pi a + \beta_k \sin k\pi a = A \bmod I, \qquad A \in \mathbb{R},$$

$$\rho^2(a) = A^2 \bmod I \in I \Longrightarrow A = 0.$$

Hence $\rho = 0$, $\Lambda = 0$ and g_{ik} in this case coincides with Minkowski metric.

3.7 Stage $\ell C^\infty(U)$

Consider stage $\ell C^\infty(U)$, where $U \subset \mathbb{R}^n$ is open set. Since

$$\ell C^\infty(U) \cong \ell C^\infty(\mathbb{R}^{n+1})/\{a_{n+1} \cdot \chi(a) - 1\},$$

$$U = \{a \in \mathbb{R}^n | \chi(a) \neq 0\}, \qquad \chi \in C^\infty(\mathbb{R}^n),$$

then with the help of transformation of variables

$$\begin{cases} a\prime = a \\ a'_{n+1} = a_{n+1} \cdot \chi(a) - 1 \end{cases}$$

we can get, for example, that

$$\rho(a, a_{n+1}) \operatorname{mod} I = \sum_{|\alpha|=1}^{\infty} A_\alpha a^\alpha, \qquad a \in U, \quad A_\alpha \in \mathbb{R}$$

$$\rho(0, \frac{1}{a(0)}) = 0.$$

4 Transitions between stages

Change of stage ℓA on stage ℓB is morphism between two stages

$$\ell B \xrightarrow{\psi} \ell A$$

then transition between $Hom_{\mathbf{L}}(\ell A, T)$ $Hom_{\mathbf{L}}(\ell B, T)$ is realized by means of

$$Hom_{\mathbf{L}}(\ell A, T) \xrightarrow{\Psi} Hom_{\mathbf{L}}(\ell B, T)$$

for any object T of category $\mathbf{L}$, which each morphism $h : \ell A \to T$ puts in correspondence morphism $h\Psi : \ell B \to T$.

Let, now, $\ell A = \ell C^\infty(\mathbb{R}^n)/I$ and $\ell B = \ell C^\infty(\mathbb{R}^m)/J$. Then transition between stages ℓA, ℓB gives metric

$$g_{ik}(b) =$$

$$\begin{pmatrix} 1 + \dfrac{(\kappa c^2 \rho(\Psi(b)) - 2\Lambda(\Psi(b)))}{6} r^2 & 0 & 0 & 0 \\ 0 & -\dfrac{1}{1 - \dfrac{(\kappa c^2 \rho(\Psi(b)) + \Lambda(\Psi(b)))}{3} r^2} & 0 & 0 \\ 0 & 0 & -r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix}$$

modulo $J \cdot C^\infty(\mathbb{R}^m \times \mathbb{R}^4)$ instead of metric (7).

The condition of infinitesimallity for Λ, ρ and condition (6) will copied so

$$(\phi \circ \rho \circ \Psi)(b) = 0 \operatorname{mod} J, \qquad (\phi \circ \Lambda \circ \Psi)(b)) = 0 \operatorname{mod} J$$

and

$$2\Lambda(\Psi)(b))\rho(\Psi)(b)) - \kappa c^2 \rho^2(\Psi)(b)) = 0 \bmod J.$$

5 Physical notes

At first note very interesting fact: at all considered stages signature of metric g_{ik} depends of the form of functions Λ, ρ and C. For example, at stage $D = \ell C^\infty(\mathbb{R})/\{a^2\}$ $\rho \,\mathrm{mod} I = \rho_1 a$, Λ mod $I = \Lambda_1 a$, C mod $I = C_1 a$, where $\rho_1, \Lambda_1, C_1 \in \mathbb{R}$ are arbitrary real numbers (under $C = 0$ the condition (6) or (6') is valid for all $a \in \mathbb{R}$). Hence field g_{ik} is not weak with respect to five dimensions (t, r, θ, ϕ, a). More interesting situation can be observed at stages $D_n(k) = \ell J_n^k = lC_0^\infty(\mathbb{R}^n)/m^{k+1}$ and $\ell C^\infty(\mathbb{R}^2)/\{a_1 - a_2\}$.

Note also that at stage $\ell C^\infty(U)$ if U is bounded then functions ρ, Λ, C are can be taken small and signature of metric does not change.

What sence have the "hidden" parameters $a \in \mathbb{R}^n$? We are thinking that they say us about existence of the additional dimensions the number of which can be changeable. For finding of coefficients ρ_1, Λ_1, C_1 at stage $\ell C^\infty(\mathbb{R})/\{a^2\}$, $\rho_{i_1,\ldots,i_l}, \Lambda_{i_1,\ldots,i_l}, C_{i_1,\ldots,i_l}$ at stage $\ell J_n^k = lC_0^\infty(\mathbb{R}^n)/m^{k+1}$, functions $\rho(a_2), \Lambda(a_2), C_(a_2)$ at stage $\ell C^\infty(\mathbb{R}^2)/\{a_1 - a_2\}$ must be possibly used the many-dimensional Einstein equations. In other words 4-dimensional intuitionistic theory contains uncountable number of many-dimensional theories. The infinitesimal field with respect to 4-dimensional universe can be found non-weak with respect to "hidden" geometry. Intuitionistic logic implies new view about Nuture of World.

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