5. M. A. Evgrafov, Yu. V. Sidorov, M. V. Fedoryuk, M. I. Shabunin, and K. A. Bezhanov, Problems in the Theory of Analytic Functions [in Russian], Nauka, Moscow (1972).
6. V. S. Azarin, "On rays of regular growth of an entire function," Matem. Sb., 79, No. 4, 463-476 (1969).
7. S. Stoilow, Lecons sur les Principes Topologiques des Fonctions Analytiques, Gauthier-Villars, Paris (1938).
8. W. K. Hayman, "Questions of regularity connected with the Phragmen–Lindelöf principle," J. Math. Pures et Appl., 35, No. 2, 115-126 (1956).
9. N. S. Landkov, Foundations of Modern Potential Theory [in Russian], Nauka, Moscow (1966).

INVARIANT ORDERS ON THREE-DIMENSIONAL LIE GROUPS

A. K. Guts

UDC 519.46

Aleksandrov has shown [1] that an isotonic (i.e., order-preserving) homeomorphism of a commutative Lie group onto itself is an isomorphism, provided the order is not quasicylindrical. An example is given in [2] of a noncommutative Lie group for which an analogous result is valid.

The following questions are of interest in this context:

1) Does an invariant order exist for any noncommutative Lie group;

2) are the corresponding isotonic homeomorphisms automorphisms;

3) to what extent does the concept of quasicylindrical order correspond to the exceptional case not covered by a theorem of Aleksandrov's type.

In the present paper we give some partial results concerning only connected, simply connected, three-dimensional Lie groups, i.e., the universal covering groups of three-dimensional groups.

It turns out that for these groups, the existence of a global invariant order is not such a rare phenomenon, which means that the result of Aleksandrov can be extended even to noncommutative groups. However, upon doing this, the quasicylindrical orders lose their exceptional character.

Two groups remain unstudied.

1. DEFINITIONS

1.1. Let G_n be an n-dimensional Lie group. We assume that each point $x \in G_n$ is put in correspondence with a set P_x in such a way that:

1) $x \in P_x$;

2) if $y \in P_x$, then $P_y \subset P_x$;

3) if $x \neq y$, then $P_x \neq P_y$.

Then it is easy to introduce a partial order on G_n by putting $x \leq y$ if and only if $y \in P_x$. If condition 3) is not satisfied, then we speak of a pre-order.

1.2. The order is called invariant if for any elements x and y we have

$$x \cdot P_y = P_{x \cdot y}.$$

1.3. A mapping $f: G_n \to G_n$ is called isotonic if it preserves the identity, i.e., f(e) = e, and if for any $x \in G_n$ we have

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 17, No. 5, pp. 986-992, September-October, 1976. Original article submitted January 21, 1975.

$$f(P_x) = P_{f(x)},$$

that is, $x \leq y$ implies $f(x) \leq f(y)$.

1.4. Let L and H ($L \cap H = \{e\}$) be a one-parameter semigroup and an $(n-1)$-dimensional subgroup of the n-dimensional group G_n, respectively. We denote by $L(g)$ ($e \neq g$) a subset of the semigroup L homeomorphic to the unit interval [0, 1] in the real numbers, e and g corresponding to the endpoints 0 and 1, respectively, of [0, 1]. Further, by $[a, b]$ we mean a subset of G_n for which there exists $h \in G_n$ such that $h \cdot [a, b] = L(g)$ and $h \cdot a = e$, $h \cdot b = g$.

We define a mapping $d_{L(g)H}$ in the following way:

(1) $d_{L(g)H}$ is a homeomorphism of G_n onto itself;

(2) $d_{L(g)H}$ maps every coset hH onto a coset h'H by left translation, i.e.,

$$d_{L(g)H}(hH) = h'' \cdot hH;$$

(3) $d_{L(g)H}$ maps every "interval" $[a, b]$ onto another "interval" of the same type.

1.5. A set is said to be a quasicylinder $Q[L(g), H]$ if its image under $d_{L(g)H}$ coincides with the image under left translation by some element in G_n, i.e., if there exists $t \in G_n$ such that

$$d_{L(g)H}(Q[L(g), H]) = t \cdot Q[L(g), H].$$

We do not exclude the case that L(g) coincides with L, and we denote such a quasicylinder by Q[L, H].

If $L_1, \ldots, L_n$ are n distinct one-parameter semigroups in G_n, then their Cartesian product is a quasicylinder $Q[L_i, H_i]$, where H_i is an $(n-1)$-dimensional subgroup generated by one-parameter subgroups $L_1', \ldots, L_{i-1}', L_{i+1}', \ldots, L_n'$, where $L_j \subset L_j'$.

2. THREE-DIMENSIONAL LIE ALGEBRAS AND LIE GROUPS

Since we do not have a general method for solving the questions that interest us, the three-dimensional Lie groups are of special interest in that there exist only nine real nonisomorphic types of Lie algebras for them. This allows us to study each Lie group G_3 case by case in terms of its dependence on its Lie algebra.

2.1. We list the three-dimensional Lie algebras g_3 (cf. [3, p. 72]).

The solvable ones are

g_3I) $[X_iX_j] = 0$ $(i, j = 1, 2, 3)$;

g_3II) $[X_1X_2] = 0$, $[X_2X_3] = X_1$, $[X_3X_1] = 0$;

g_3III) $[X_1X_2] = 0$, $[X_2X_3] = 0$, $[X_1X_3] = X_1$;

g_3IV) $[X_1X_2] = 0$, $[X_2X_3] = X_1 + X_2$, $[X_1X_3] = X_1$;

g_3V) $[X_1X_2] = 0$, $[X_2X_3] = X_2$, $[X_1X_3] = X_1$;

g_3VI) $[X_1X_2] = 0$, $[X_2X_3] = qX_2$, $[X_1X_3] = X_1$ $(q \neq 0, 1)$;

g_3VII) $[X_1X_2] = 0$, $[X_2X_3] = -X_1 + qX_2$, $[X_1X_3] = X_2$ $(q^2 < 4)$.

The nonsolvable ones are

g_3VIII) $[X_1X_2] = X_1$, $[X_2X_3] = X_3$, $[X_1X_3] = 2X_2$;

g_3IX) $[X_1X_2] = X_3$, $[X_2X_3] = X_1$, $[X_3X_1] = X_2$.

2.2. We are interested only in connected, simply connected Lie groups. For a Lie algebra g_3 there exists a global connected, simply connected Lie group G_3 having Lie algebra isomorphic to g_3 [5]. Moreover, in the case of a solvable Lie algebra the corresponding group is homeomorphic to the Euclidean space E_3 and hence is noncompact [4, p. 432].

Since all Lie groups with a given Lie algebra are locally isomorphic [5], the connected, simply connected Lie group appearing among them is unique up to isomorphism [4, p. 374]. This Lie group is the universal covering group.

The connected, simply connected Lie groups with Lie algebras g_3I$-g_3$IX will be denoted by G_3I$-G_3$IX, respectively.

2.3. The following assertion is evident: If the group G has an invariant order, and if the group G' is isomorphic to G, i.e., $G \cong G'$, then G' also has an invariant order.

3. ORDERS ON THE LIE GROUPS $G_3I - G_3VI$ AND G_3IX

3.1. We say that an order P_e is good if

1) P_e contains an interior point;

2) $(\exp)_e^{-1}(P_e)$ contains a ray issuing from zero.

We are interested only in good orders.

3.2. We say that the group G has property $\mathscr{A}$, if it has a good invariant order such that any isotonic homeomorphism of G onto itself is an automorphism.

3.3. THEOREM. The Lie groups $G_3I - G_3V$ and G_3VI $(0 < q < 1)$ have property $\mathscr{A}$. The group G_3IX does not not have a good order.

Proof. We represent the group G_3 as a group of transformations acting simply transitively on some sufficiently good space M. M will be either the Euclidean space E_3, the hyperbolic space $Л_3$, or the sphere S^3.

Fix a point $x_0 \in M$. Then we easily get a homeomorphism φ between M and G_3

$$G_3 \ni g \overset{\varphi}{\to} g(x_0) \in M. \tag{1}$$

If now P is an invariant order on G_3, the family $\{P'_x : x \in M\}$, where $P'_x = \varphi(P_{\varphi^{-1}(x)})$ defines an order on M invariant with respect to G_3, i.e., for any $x \in M$ and $g \in G_3$ we have $g(P'_x) = P'_{g(x)}$. Conversely, if $\{P'_x\}$ is an order on M invariant with respect to G_3, then $\{P_g : g \in G_3\}$, where

$$P_g = \varphi^{-1}(P'_{\varphi(g)})$$

is an invariant order on G_3.

In view of this, it suffices to study the orders on M invariant with respect to G_3 (the G_3-invariant orders).

a) G_3I. Here $M = E_3$ and G_3I is a group of translations. That G_3I has property $\mathscr{A}$ follows from the results of Aleksandrov [1].

b) G_3II. Here $M = Л_3$. It is convenient to pass to the Poincaré model $\hat{Л}_3$ of $Л_3$: $\hat{Л}_3 = \{(x, y, z) \in E_3 : z > 0\}$, where x, y, z are rectilinear Cartesian coordinates. The group G_3II consists of the transformations of the type

$$g: (x, y, z) \to (x+\alpha, \lambda x + y + \beta, (1+\lambda)z),$$

where $\lambda > -1$, α, β are real numbers. The infinitesimal operators are:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

A G_3II-invariant order is defined by the quasicylinder

$$P_{(0,0,\alpha)} = \{(x, y, z) \in \hat{Л}_3 : x \geqslant 0,\ y \geqslant (\alpha - 1)x,\ z \geqslant \alpha\},\ x_0 = (0, 0, 1),$$

and P_x, P_y are obtained from one another by parallel translation in case their z coordinates are equal. We show that an isotonic homeomorphism f $(f(x_0) = x_0)$ for the above order must have the form

$$f(x, y, z) = (\mu x, \mu y, z). \tag{2}$$

Denote by $\Gamma^1_{x_0}$, $\Gamma^2_{x_0}$, $\Gamma^3_{x_0}$, respectively, the faces of the three-faced angle ∂P_{x_0} which lie in the planes given by $z = 1$, $x = 0$, $y = 0$. Put $\Gamma^i_x = g(\Gamma^i_{x_0})$, where $g \in G_3$ is such that $g(x_0) = x$.

Let

$$\Pi^1_\lambda = \bigcup_{(x,y)} \Gamma^1_{(x,y,\lambda)},\ \Pi^2_\alpha = \bigcup_{(y,z)} \Gamma^2_{(\alpha,y,z)},\quad \Pi^3_{\beta\gamma} = \bigcup_{y=(\beta-1)x} \Gamma^3_{(x,y,\beta)} + \overrightarrow{(0,\gamma,0)},\ \beta > 0.$$

Clearly, Π^1_λ is the plane $\{z = \lambda\}$, Π^2_α is the half-plane $\{x = \alpha,\ z > 0\}$, and $\Pi^3_{\beta\gamma}$ is the half-plane $\{y = (\beta - 1)x,\ z \geq \beta\}$. Since f is a homeomorphism and $\Pi^3_{\beta\gamma}$ has boundary, the image of $\Pi^3_{\beta\gamma}$ can only be $\Pi^3_{\beta'\gamma'}$. Since for any Π^1_λ there exists $\Pi^3_{\beta\gamma}$ such that $\Pi^1_\lambda \cap \Pi^3_{\beta\gamma} = \emptyset$, the image of Π^1_λ can only be $\Pi^1_{\lambda'}$, for any Π^2_α intersects any $\Pi^3_{\beta\gamma}$. It follows from this that f preserves the family of the z coordinates of the lines and also the coordinate plane $\{z = 1\}$. But then f has the form

$$f(x, y, z)=(f_1(x, y), f_2(x, y), f_3(z)).$$

We remark that if $\beta_1, \beta_2, \beta_3$ are distinct and $\beta_1, \beta_2, \beta_3 < 1$, then the intersections $\Pi_1^1 \cap \Pi^3_{\beta_1,\gamma}$, $\Pi_1^1 \cap \Pi^3_{\beta_2\gamma}$, $\Pi_1^1 \cap \Pi^3_{\beta_3\gamma}$ define three families of parallel lines on $\Pi_1^1 = \{z = 1\}$. These are mapped by f into three families of parallel lines. But then f is affine on Π_1^1, i.e.,

$$f_1(x, y)=a \cdot x+b \cdot y, \quad f_2(x, y)=cx+dy$$

[1, p. 12].

Since the lines $\{x = 0, z = 1\}$ and $\{y = 0, z = 1\}$ go into themselves, we have $b = c = 0$, i.e., $f_1(x, y) = ax$, $f_2(x, y) = dy$.

The numbers a, d are positive since $f(P_{x_0}) = P_{x_0}$.

Since $f(P_{(0,0,\lambda)}) = P_{(0,0,(d/a)\lambda)} = P_{(0,0,f_3(\lambda))}$, we have $f_3(\lambda) = d\lambda/a$.

But $f_3(1) = 1$. Hence $d = a = \mu$ and $f_3(z) = z$. This proves Eq. (2). An isotonic homeomorphism $f: Л_3 \to Л_3$ induces an isotonic homeomorphism $\tilde{f}: G_3II \to G_3II$ defined by (1), i.e.,

$$\tilde{f}(g) \xrightarrow{\varphi} f(g(x_0)). \tag{3}$$

The converse is also valid. Let us show that $\tilde{f}$ is an automorphism.

Let $g_1, g_2 \in G_3II$ and

$$g_1(x, y, z)=(x+\alpha_1, \lambda_1 x+y+\beta_1, (1+\lambda_1)z), \quad g_1 \xrightarrow{\varphi} (\alpha_1, \beta_1, (1+\lambda_1)),$$
$$g_2(x, y, z)=(x+\alpha_2, \lambda_2 x+y+\beta_2, (1+\lambda_2)z), \quad g_2 \xrightarrow{\varphi} (\alpha_2, \beta_2, (1+\lambda_2)).$$

Then

$$g_2 g_1 \xrightarrow{\varphi} g_2(g_1(x_0))=(\alpha_1+\alpha_2, \lambda_2\alpha_1+\beta_1+\beta_2, (1+\lambda_1)(1+\lambda_2)) \tag{4}$$

defines the law of multiplication in the group G_3II.

Further,

$$\tilde{f}(g_2 g_1) \xrightarrow{\varphi} f(g_2 g_1(x_0))=(\mu\alpha_1+\mu\alpha_2, \mu\lambda_2\alpha_1+\mu\beta_1+\mu\beta_2, (1+\lambda_1)(1+\lambda_2)), \tag{5}$$
$$\tilde{f}(g_1) \xrightarrow{\varphi} f(g_1(x_0))=(\mu\alpha_1, \mu\beta_1, (1+\lambda_1)),$$
$$\tilde{f}(g_2) \xrightarrow{\varphi} f(g_2(x_0))=(\mu\alpha_2, \mu\beta_2, (1+\lambda_2)).$$

Multiplying the elements $\tilde{f}(g_1)$ and $\tilde{f}(g_2)$ by (4) we get

$$\tilde{f}(g_2)\tilde{f}(g_1) \xrightarrow{\varphi} (\mu\alpha_1+\mu\alpha_2, \mu\lambda_2\alpha_1+\mu\beta_1+\mu\beta_2, (1+\lambda_1)(1+\lambda_2)). \tag{6}$$

Comparing (5) and (6) we conclude that

$$\tilde{f}(g_2)\tilde{f}(g_1)=\tilde{f}(g_2 g_1),$$

i.e., $\tilde{f}$ is an automorphism.

This means that G_3II has property $\mathscr{A}$.

c) G_3III. We take $M = Л_3$, where G_3III is given in $Л_3$ by transformations of the type

$$g: (x, y, z) \to (x+\alpha, \lambda y+\beta, \lambda z), \quad x_0=(0, 0, 1),$$

where $\lambda, z > 0$.

Here, the order is the following:

$$P_{(0,0,\lambda)}=\{(x, y, z) \in \hat{Л}_3 : x \geqslant 0, \ y \geqslant \lambda x, \ z \geqslant \lambda\}, \ \lambda>0, \tag{7}$$

and P_x, P_y are equal and parallel for x and y with equal z coordinate, i.e., for $z(x) = z(y)$.

Denote by $\Gamma^1_{x_0}$, $\Gamma^2_{x_0}$, $\Gamma^3_{x_0}$, respectively, the faces of the three-faced angle ∂P_{x_0} which lie in the planes defined by $z = 1$, $x = 0$, and $y = x$. Put $\Gamma^i_x = g(\Gamma^i_{x_0})$, where $g \in G_3$ is such that $g(x_0) = x$.

Let

$$\Pi_\lambda^1 = \bigcup_{(x,y)} \Gamma^1_{(x,y,\lambda)}, \quad \Pi_\alpha^2 = \bigcup_{(y,z)} \Gamma^2_{(\alpha,y,z)},$$

$$\Pi^3_{\beta\gamma} = \bigcup_{y=\beta x} \Gamma^3_{(x,y,\beta)} + \overrightarrow{(0, \gamma, 0)}, \quad \beta > 0.$$

Evidently, $\Pi_\lambda^1 = \{z = \lambda\}$, $\Pi_\alpha^2 = \{x = \alpha,\ z > 0\}$, and $\Pi^3_{\beta 0} = \{y = \beta x,\ z \geq \beta\}$. Then we see as in b) that f has the form

$$f(x, y, z) = (ax+by,\ cx+dy,\ f_3(z)).$$

Since the lines $\{x = 0,\ z = 1\}$, $\{y = x,\ z = 1\}$ go into themselves, we have $b = 0$ and $d + c = a$. We see again as in b) that

$$f_3(z) = (zd+c)/a.$$

Further, the ray $L = \{y = 0,\ z = 1,\ x \geq 0\}$ is mapped into the ray L' lying in $\{z = 1\}$ and starting at the point x_0.

But L is a limit of the rays

$$L_n = \Pi^3_{1/n,0} \cap \{z=1\} \cap \{x \geqslant 0\} \quad \text{as} \quad n \to \infty.$$

Since f is a homeomorphism and $(\{x \geq 0\} \cap \{y > 0\}) \cup \{x = 0,\ y = 0\} = \bigcup_{\lambda>0} P_{(0,0,\lambda)}$, we have that L' is a limit of the rays

$$\Pi^3_{f_3(1/n),0} \cap \{z=1\} \cap \{x \geqslant 0\} \text{ and } f_3(1/n) \xrightarrow[n\to\infty]{} 0.$$

It follows from this that L' = L. But then, $c = 0$, $a = d = \mu$, i.e., f has the form (2). It is trivial to check that $\tilde{f}$ is an automorphism.

The above order is quasicylindrical. So also, e.g., is the order given by:

$$P_{(0,0,\lambda)} = \{(x, y, z) \in \hat{\Pi}_3 : z \geqslant \lambda,\ x \geqslant 0,\ y \geqslant 0\}, \tag{8}$$

and P_x and P_y are equal and parallel for $z(x) = z(y)$. This order is preserved by maps of the type $f(x, y, z) = (f_1(x), f_2(y), f_3(z))$, where f_i are in general arbitrary functions. These functions may, moreover, be chosen so that $\tilde{f}$ is not an automorphism [for instance, if $f_3(\alpha\beta) \neq f_3(\alpha)f_3(\beta)$].

There exists an invariant order on G_3III which is not quasicylindrical, namely,

$$P_{(\alpha,\beta,\lambda)} = \{(x, y, z) \in \hat{\Pi}^3 : \lambda^2 x^2 + y^2 - (z-\lambda)^2 \leqslant 0,\ z \geqslant \lambda\} + \overrightarrow{(\alpha, \beta, 0)}.$$

One might naturally suppose that $\tilde{f}$ is an automorphism also with respect to this order, but this fact is not necessary for our purposes.

d) G_3IV. Put $M = \Pi_3$, with the group given on $\hat{\Pi}_3$ by the transformations

$$g: (x, y, z) \to ((1+\lambda)x + \lambda y + \alpha,\ (1+\lambda)y + \beta,\ (1+\lambda)z),\ \lambda > -1,\ z > 0.$$

The invariant order here is the same as the one in b).* The rest of the proof is analogous to b).

e) G_3V. $M = \Pi_3$ with G_3V of the form

$$g: (x, y, z) \to (\lambda x + \alpha,\ \lambda y + \beta,\ \lambda z).$$

As shown in [2], it has property $\mathscr{A}$. This group and G_3I have been more thoroughly studied than the others.

f) G_3VI $(0 < q < 1)$. $M = \Pi_3$, with G_3VI given in $\hat{\Pi}_3$ by

$$g: (x, y, z) \to ((1+\lambda)x + \alpha,\ (1+\lambda q)y + \beta,\ (1+\lambda)z),$$

where $\lambda > -1$, $z > 0$.

The invariant order with respect to which G_3VI has property $\mathscr{A}$ is:

$$P_{(0,0,\lambda)} = \{(x, y, z) \in \hat{\Pi}_3 : y \geqslant 0,\ x \geqslant [\lambda/(1+(\lambda-1)q]y,\ z \geqslant \lambda\},\ \lambda > 0,$$

and P_x, P_y are equal and parallel for $z(x) = z(y)$.

---

*More precisely, $P_{(0,0,\alpha)} = \{y \geq 0,\ x \geq (\alpha - 1)y/\alpha,\ z \geq \alpha\}$, $\alpha > 0$.

We establish as in c) that an isotonic homeomorphism has the form (2). It is trivial to check that $\tilde{f}: G_3 \to G_3$ is an automorphism.

g) G_3IX. The Lie algebra of the group G_3IX is semisimple and compact. Therefore, G_3IX is compact [4, pp. 446, 483]. But then $G_3IX \cong SU(2)$ [4, p. 498, E]. Since $(\exp)_e^{-1}(P)$ contains a ray, it follows that P contains a one-parameter semigroup $\gamma(t)$, $t \in [0, +\infty)$. Let $\Gamma(t)$, $t \in (-\infty, +\infty)$, be a one-parameter subgroup containing $\gamma(t)$. The closure $\overline{\Gamma(t)}$ of $\Gamma(t)$ is an Abelian subgroup of the Lie group G_3IX, and is moreover compact and connected. This means that $\overline{\Gamma(t)}$ is a torus [6]. Since a maximal torus of the group SU(2) is one-dimensional [6], $\Gamma(t)$ is a compact curve. Then we can find $a = \Gamma(t_0)$ such that $a \neq e$, $e \leq a$, and $a \leq e$. But this contradicts the third axiom for an order (cf. Sec. 1). This proves the theorem.

4. CONCLUSIONS

At the beginning of this paper we stated three questions about orders on Lie groups. Now we can answer them in the following way.

1) Global invariant orders exist on many noncommutative Lie groups.

2) A theorem of the type of Aleksandrov's theorem [1] holds for the group G_3V [2], but for the others, it cannot be formulated in the same way as was done in [1]. Therefore, we speak only of these groups as having property $\mathscr{A}$.

3) A basic difficulty is caused by the fact that for the groups G_3II-G_3IV and G_3VI the concept of quasicylindrical order does not lead to the exceptional case as was the situation with G_3I and G_3V [1, 2]. This is shown by the existence of quasicylindrical orders on G_3III such as (7) and (8).

4) In studying the groups G_3I-G_3VI, we see that if they have one-parameter semigroups L_1, L_2, L_3 such that: (a) $P = L_1 \times L_2 \times L_3$ is a quasicylindrical order; (b) for any $g \in L_3'$ we have $g(L_1' \times L_3') = L_1' \times L_3'$, $g(L_2' \times L_3') = L_2' \times L_3'$, where L_i' is a one-parameter subgroup containing L_i ($i = 1, 2, 3$); (c) $L_1' \times L_2'$ is an Abelian subgroup, then an isotonic map P need not necessarily be an automorphism.

Properties (b) and (c) are precisely the ones distinguishing the orders (7), (8). It seems possible that the exceptional case in Aleksandrov's theorem can in fact be reduced to the existence of a quasicylinder with properties (a), (b), and (c). And then it seems entirely possible that any isotonic mapping in the group G_3II might be an automorphism (for a good order).

We remark in conclusion that one can show using the above method the existence of four-dimensional Lie groups having property $\mathscr{A}$. For this, one uses the classification of real four-dimensional Lie algebras [3].

LITERATURE CITED

1. A. D. Aleksandrov, "Mappings of ordered spaces," Trudy Matem. In-ta Akad. Nauk SSSR, 128, 3-21 (1972).
2. A. K. Guts, "Mappings of an ordered Lobachevskii space," Dokl. Akad. Nauk SSSR, 215, No. 1, 35-37 (1974).
3. A. Z. Petrov, New Methods in the General Theory of Relativity [in Russian], Nauka, Moscow (1966).
4. L. S. Pontryagin, Continuous Groups [in Russian], Nauka, Moscow (1973).
5. J.-P. Serre, Lie Algebras and Lie Groups, Benjamin, New York (1965).
6. D. Husemoller, Fiber Bundles, McGraw-Hill, New York (1962).