CHRONOGEOMETRY OF EINSTEIN SPACES OF MAXIMAL MOBILITY

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In this article we give necessary and sufficient conditions which assure that a bijective map of a Lorentz manifold ($\overset{*}{T}_{2,5}$ and $\overset{*}{T}_{3,4}$, four-dimensional Einstein spaces) is a motion. The main point here is that we do not require the map to be differentiable (this condition is needed in the classical definition of a motion).

We consider a four-dimensional Lorentz manifold $\langle V, g\rangle$ which is elementary, i.e., diffeomorphic to Euclidean space R^4. We can introduce global coordinates x^1, x^2, x^3, x^4 on V in terms of which the Lorentz metric g is given by the differential form

$$ds^2 = \sum_{i,k} g_{ik}(x^1, x^2, x^3, x^4)\, dx^i dx^k.$$

An isotropic cone $\tilde{C}_u$ at a point $u \in V$ is a cone contained in the tangent space V_u at u such that every vector ξ in $\tilde{C}_u$ satisfies the equation

$$g_u(\xi, \xi) = 0,$$

or, in coordinate form,

$$\sum_{i,k} g_{ik}(u^1, u^2, u^3, u^4)\, \xi^i \xi^k = 0,$$

where $u = (u^1, u^2, u^3, u^4)$.

To the cone $\tilde{C}_u$ we associate a subset C_u of the manifold V as follows. A point $x \in C_u$ if and only if x satisfies the equation

$$\sum_{i,k} g_{ik}(u^1, u^2, u^3, u^4)(x^i - u^i)(x^k - u^k) = 0. \tag{1}$$

Let $f: V \to V$ be a motion of V. Then f preserves isotropic cones, i.e.,

$$(df)_u(\tilde{C}_u) = \tilde{C}_{f(u)} \tag{2}$$

for every point $u \in V$. It is easy to see that if f has the property that

$$f(C_u) = C_{f(u)} \tag{3}$$

for every point $u \in V$, then f satisfies (2). The converse, in general, is false.

However, we do have the following lemma.

LEMMA. If G_r is an r-parameter group of motions of the manifold $\langle V, g\rangle$ which is affine in the global coordinates $\{x^i\}$, i.e., each motion $f \in G_r$ is an affine transformation, then Eqs. (2) and (3) imply one another.

The following question arises naturally: Could not one define a Lorentz motion of a manifold as a bijective map $f: V \to V$ satisfying condition (3)? Such an approach makes it possible to remove the requirement that f be differentiable and involves only preservation of zero lengths, and not preservation of derivatives (as is customary in defining motions).

However, it is easily seen that this approach to defining motions extends only to those homogeneous Lorentz manifolds which admit coordinates with respect to which the group of motions is affine. Moreover, the family of "cones" $\{C_u : u \in V\}$ is constructed from a family of isotropic cones $\{\tilde{C}_u : u \in V\}$, but this construction is covariant only with respect to the group of affine coordinate transformations. Therefore, our approach is closely related with the problem of choosing privileged coordinate systems with respect to which the group of motions is affine. The existence of privileged coordinate systems depends on the properties of the manifold

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in the large [1, p. 12], and for this reason our approach is based on deeper principles than might appear at first glance. This remark takes on physical meaning if we recall that in and of itself, the principle of general covariance is not the expression of any physical law [1, p. 13], whereas the choice of privileged coordinate systems is related to the attempt to express physical laws in their simplest form [2, p. 122].

Finally, since we have to establish the differentiability of the map f considered, it suffices for this to choose a single coordinate system from an atlas of charts.

Definition. Assume that a group of motions of a manifold V becomes affine in the coordinates $\{x^i\}$. In this case, a map preserving isotropic cones is understood to be a bijection $f: V \to V$ satisfying condition (3).

In this article, we study bijections which preserve isotropic cones in Einstein spaces of maximal mobility satisfying the equations

$$R_{ik} = \varkappa g_{ik}$$

for some constant $\varkappa \neq 0$.

Such manifolds are known to belong to one of three types described by Petrov [3, p. 249]:

1) the space $\overset{*}{T}_1$, a space of constant curvature;

2) $\overset{*}{T}_{2,5}$, a space of Petrov type II with a five-member transitive group of motions;

3) $\overset{*}{T}_{3,4}$, a space of Petrov type III with a four-member transitive group of motions.

We studied the first case (a De Sitter manifold) in [8] without considering the global topological structure. Therefore, in this work we only study the other two cases, the spaces $\overset{*}{T}_{2,5}$ and $\overset{*}{T}_{3,4}$. Topologically, $\overset{*}{T}_{2,5}$ and $\overset{*}{T}_{3,4}$ are homeomorphic to R^4 [3] and they are therefore elementary.

In coordinates with respect to which the group of motions is affine, $\overset{*}{T}_{2,5}$ is defined as follows [4, p. 214]:

$$ds^2 = -e^{2\omega x^4}\left(2dx^1dx^3 + dx^{2^2}\right) + \varepsilon e^{-\omega x^4}dx^{3^2} - dx^{4^2} \tag{4}$$

with motion group G_5

$$\left.\begin{aligned} \bar{x}^1 &= e^{5\alpha_5}x^1 - \alpha_4 e^{5\alpha_5}x^2 - \frac{\alpha_4^2}{2}e^{5\alpha_5}x^3 + \alpha_1 \\ \bar{x}^2 &= e^{2\alpha_5}x^2 + \alpha_4 e^{2\alpha_5}x^3 + \alpha_2 \\ \bar{x}^3 &= e^{-\alpha_5}x^3 + \alpha_3 \\ \bar{x}^4 &= x^4 - \frac{2}{\omega}\alpha_5, \end{aligned}\right\} \tag{5}$$

where $\omega = \sqrt{\varkappa/3}\,(\varkappa > 0)$, $\varepsilon = \pm 1$ and α_i $(i = 1, 2, 3, 4, 5)$ are arbitrary real parameters.

For the space $\overset{*}{T}_{3,4}$ we have correspondingly [4, p. 218]:

$$\begin{gathered} ds^2 = e^{-2\omega x^4}\left(2\varepsilon_1 dx^1dx^3 - dx^{2^2}\right) + 2\varepsilon_2 e^{\omega x^4}dx^2dx^3 - \frac{1}{2}e^{4\omega x^4}dx^{3^2} - dx^{4^2}, \\ \omega = \sqrt{\varkappa/3},\quad \varepsilon_1 = \pm 1,\quad \varepsilon_2 = \pm 1 \end{gathered} \tag{6}$$

and the group G_4 is given by

$$\left.\begin{aligned} \bar{x}_1 &= e^{4\beta}x^1 + \alpha \\ \bar{x}_2 &= e^{\beta}x^2 + \gamma \\ \bar{x}_3 &= e^{-2\beta}x^3 + \delta \\ \bar{x}_4 &= x^4 + \beta/\omega \end{aligned}\right\}, \tag{7}$$

where α, β, γ, δ are arbitrary real parameters.

We now state the main result of this work.

THEOREM. Any bijective map of the spaces $\overset{*}{T}_{2,5}$ or $\overset{*}{T}_{3,4}$ onto themselves which preserves isotropic cones is a motion.

This theorem should be regarded as a generalization of a theorem of Aleksandrov and Ovchinnikova [5] to the case of twisted Lorentz manifolds.

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1. Proof of the Lemma

(1.1) Let

$$\bar{x}^i = \bar{x}^i(x^1, x^2, x^3, x^4) \equiv \sum_{k=1} a^i_k x^k + \alpha^i \tag{8}$$

be a transformation of the manifold V belonging to the group of motions. If the map in (8) has property (2), then

$$g_{ik}(\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{u}^4)d\bar{x}^i d\bar{x}^k = g_{ik}(u^1, u^2, u^3, u^4)dx^i dx^k = 0 \tag{9}$$

or

$$g_{ik}(\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{u}^4)\, a^i_n a^k_m = g_{nm}(u^1, u^2, u^3, u^4),$$

where $(\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{u}^4)$ and (u^1, u^2, u^3, u^4) are the coordinates of the point $u \in V$ after and before application of the motion corresponding to (8). Consequently,

$$0 = g_{nm}(u^1, u^2, u^3, u^4)(x^n - u^n)(x^m - u^m) = g_{ik}(\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{u}^4)\, a^i_n a^k_m \times$$
$$\times (x^n - u^n)(x^m - u^m) = g_{ik}(\bar{u}^1, \bar{u}^2, \bar{u}^3 \bar{u}^4)(\bar{x}^i - \bar{u}^i)(\bar{x}^k - \bar{u}^k), \tag{10}$$

i.e., we obtain Eq. (3). Consequently, assume Eq. (3) is satisfied, i.e., (10) holds. Then (10) immediately implies (9) or (2). The lemma is proved.

(1.2) Proposition. Every conformal transformation of the space $\overset{*}{T}_{2,5}$ or $\overset{*}{T}_{3,4}$ of the form

$$\bar{x}^i = \sum_{k=1}^{4} a^i_k x^k + \alpha^i \tag{11}$$

is trivial, i.e., a motion.

In fact, the Killing vectors for conformal transformations of the type (11) have the form

$$\xi^i = \sum_{k=1}^{4} b^i_k x^k + \beta^i.$$

These Killing vectors are known to satisfy the equations

$$\xi^n \frac{\partial g_{ik}}{\partial x^n} + g_{in}\frac{\partial \xi^n}{\partial x^k} + g_{kn}\frac{\partial \xi^n}{\partial x^i} = \lambda g_{ik}, \tag{12}$$

where λ is some function.

Consider the space $\overset{*}{T}_{2,5}$. Writing out the Killing equations (12) for pairs (ik) equal to (22) and (44), we obtain

$$-2\omega\xi^4 e^{2\omega x^4} - 2e^{2\omega x^4} b^2_2 = -\lambda e^{2\omega x^4},$$
$$-2b^4_4 = -\lambda,$$

from which it is clear that

$$\lambda = 2b^4_4 = \text{const},$$
$$\lambda = 2\omega\xi^4 + 2b^2_2.$$

Consequently, $\xi^4 = \text{const}$, i.e., $b^4_1 = b^4_2 = b^4_3 = b^4_4 = 0$. But the last equations give $\lambda = 0$. Thus, the assertion in (1.2) is valid for the space $\overset{*}{T}_{2,5}$.

Now consider the space $\overset{*}{T}_{3,4}$ and write out Eqs. (12) for (ik) equal to (22) and (44):

$$2\omega\xi^4 e^{-2\omega x^4} - 2e^{-2\omega x^4} b^2_2 + 2\varepsilon_2 e^{\omega x^4} b^3_2 = -\lambda e^{-2\omega x^4},$$
$$-2b^4_4 = -\lambda.$$

Then

$$\lambda = 2b^4_4 = \text{const},$$
$$b^3_2 = 0,$$

$$\lambda = 2b_2^2 - 2\omega\xi^4.$$

This gives $b_1^4 = b_2^4 = b_3^4 = b_4^4 = 0$ and $\lambda = 0$, i.e., a conformal transformation of type (11) is a motion.

Assertion (1.2) is proved.

2. Proof of the Theorem

A) Take the case $\overset{*}{T}_{2,5}$.

The cone C_u for $\overset{*}{T}_{2,5}$ is given by the equation

$$-2e^{2\omega u^4}(x^1-u^1)(x^3-u^3) - e^{2\omega u^4}(x^2-u^2)^2 + \varepsilon e^{-\omega u^4}(x^3-u^3)^2 - (x^4-u^4)^2 = 0. \tag{13}$$

In what follows we identify V and R^4, and the terms "hyperplane," "line," "parallel," etc. refer to R^4 and not to the Riemannian structure on V.

We write $H^4_{(a)}$ for the hyperplane in V defined by the equations $x^4 = a = \text{const}$. In each hyperplane $H^4_{(u_0^4)}$ we have a family of lines and parallel two-dimensional cones:

$$S(u^1, u^2, u^3) = \left\{(x^1, x^2, x^3)\colon -2e^{2\omega u_0^4}(x^1-u^1)(x^3-u^3) - e^{2\omega u_0^4}(x^2-u^2)^2 + \varepsilon e^{-\omega u_0^4}(x^3-u^3)^2 = 0\right\},$$

where $u = (u^1, u^2, u^3, u_0^4) \in H^4_{(u_0^4)}$.

Let l_u be an arbitrary generator of the cone $S(u^1, u^2, u^3)$, where $u = (u^1, u^2, u^3, u_0^4)$. We now prove that the image of l_u is a line $f(l_u)$ lying in the hyperplane $H^4_{[f^4(u)]}$. For the points $v, w \in l_u$, $(v \neq w, v \neq u, u \neq w)$, we have

$$S(u^1, u^2, u^3) \cap S(v^1, v^2, v^3) \cap S(w^1, w^2, w^3) = l_u = l_v = l_w = C_u \cap C_v = C_u \cap C_w = C_v \cap C_w. \tag{14}$$

Assume that the point f(v) does not lie in the hyperplane $H^4_{[f^4(u)]}$. Since the cone $K \equiv C_{f(v)} \cap H^4_{[f^4(v)]}$ is distinct from and not parallel to the asymptotic cone of the two-sheeted hyperboloid $\Gamma \equiv C_{f(u)} \cap H^4_{[f^4(v)]}$, there exists a generator L of the cone K intersecting Γ at a point x distinct from f(v). Since $z \in C_{f(v)} \cap C_{f(u)}$, we have $f^{-1}(z) \in C_v \cap C_u = l_u$. But by (14) we obtain [setting $w = f^{-1}(z)$]:

$$C_{f^{-1}(z)} \cap C_v = C_{f^{-1}(z)} \cap C_u,$$

or

$$C_z \cap C_{f(v)} = C_z \cap C_{f(u)}.$$

However, $L = C_z \cap C_{f(v)}$ and $f(u) \in C_z \cap C_{f(u)} = L$. But this is impossible, since L is contained in $H^4_{[f^4(v)]}$, which does not contain the point f(u).

Consequently, the above contradiction shows that (14) implies $f(v) \in H^4_{[f^4(u)]}$. This implies that the line l_u is mapped onto the line $f(l_u) = C_{f(u)} \cap C_{f(v)}$. Thus, all the generators of the cone $f\left(C_u \cap H^4_{(u_0^4)}\right)$ lie in $H^4_{[f^4(u)]}$. Therefore, f maps the family

$$\left\{C_u \cap H^4_{(u_0^4)}\colon u \in H^4_{(u_0^4)}\right\}$$

of equal and parallel cones onto the similar family of cones $\left\{C_w \cap H^4_{[f^4(u)]}: w \in H^4_{[f^4(u)]}\right\}$. But then by Theorem 3 in [6], f is affine on $H^4_{(u_0^4)}$, and in particular,

$$f\left(H^4_{(u_0^4)}\right) = H^4_{[f^4(u)]}.$$

Let l be a line lying in $H^4_{(u_0^4)}$ passing through the point u, and not intersecting the set

$$-2e^{2\omega u_0^4}(x^1-u^1)(x^3-u^3) - e^{2\omega u_0^4}(x^2-u^2)^2 + \varepsilon e^{-\omega u_0^4}(x^3-u^3)^2 - (x^4-u_0^4)^2 \geqslant 0, \tag{15}$$

except for the point u. The set*

$$Q_u \overset{df}{\equiv} C_u \setminus \bigcup_{\substack{x \in l \\ x \neq u}} (C_x \cap C_u) \tag{16}$$

*This construction was shown to us in the three-dimensional case by A. V. Kuz'minykh.

is a two-dimensional cone which "sticks out" in the fourth dimension from the hyperplane $H^4_{(u_0^4)}$. Now let l' be a line distinct from l and not intersecting the set in (15) except for the point u and contained in the intersection of $H^4_{(u_0^4)}$ with a three-plane containing the cone Q_u. We denote by Q_x the cone obtained from Q_u by a shift taking u into x. Then the set

$$Q_u \diagdown \bigcup_{\substack{x \in l' \\ x \neq u}} (Q_x \cap Q_u) \tag{17}$$

is a pair of intersecting lines which "stick out" from the hyperplane $H^4_{(u_0^4)}$. We denote these lines by l_u^1 and l_u^2, respectively. They span a two-dimensional plane T which intersects each hyperplane $H^4_{(a)}$ in a line $l^3_{(a)}$. If the point u is moved along $T \cap H^4_{(u_0^4)}$, we obtain two families of pairwise parallel lines $\left\{l_v^1,\ l_v^2\colon v \in T \cap H^4_{(u_0^4)}\right\}$. But the lines $l^3_{(a)}$ for arbitrary a define a third family of parallel lines on T, which we denote by $\{l_v^3 \colon v \in T\}$. It is now easy to see that since f is affine on each hyperplane $H^4_{(a)}$ (since the number u_0^4 is arbitrary) and preserves the cones $\{C_u\}$, f preserves the constructions in (16), (17) no matter which point $u \in H^4_{(u_0^4)}$ we choose. Hence f maps the two-plane T onto the two-plane f(T) and takes the three families of parallel lines $\{l_u^i \colon u \in T\}$ (i = 1, 2, 3) onto three similar families of parallel lines. But then f is affine on T [7].

We choose three lines $L^A_{u_0}$ (A = 1, 2, 3) in the hyperplane $H^4_{(u_0^4)}$ and one line $L^4_{u_0}$ contained in T such that the lines $L^A_{u_0}$ (A = 1, 2, 3, 4) are in general position, i.e., their directions are linearly independent in R^4. There exists an affine map g: $R^4 \xrightarrow{\text{onto}} R^4$ with the property

$$g(f(u_0)) = u_0, \quad g\left[f\left(L^A_{u_0}\right)\right] = L^A_{u_0} \ (A = 1, 2, 3, 4).$$

But then the map $g \circ f$ is affine onto $H^4_{(u_0^4)}$ and T, and $(g \circ f)(H^4_{(u_0^4)}) = H^4_{(u_0^4)}$, $(g \circ f)(T) = T$ and $(g \circ f)(L^A_{u_0}) = L^A_{u_0}$ (A = 1, 2, 3, 4). We choose the lines $L^A_{u_0}$ (A = 1, 2, 3, 4) as new coordinate axes. It is easy to see that $g \circ f$ is defined by an affine transformation in these coordinates. Consequently, f is affine from R^4 onto R^4, i.e.,

$$f^i(x) = \sum_{k=1}^{4} a_k^i x^k + a^k. \tag{18}$$

Since

$$f(C_u) = C_{f(u)},$$

we have along with (13) that

$$\begin{aligned} &-2e^{2\omega f^4(u)} \cdot [f^1(x) - f^1(u)]\,[f^3(x) - f^3(u)] - e^{2\omega f^4(u)} \cdot [f^2(x) - f^2(u)]^2 + \\ &\quad + \varepsilon e^{-\omega f^4(u)} \cdot [f^3(x) - f^3(u)]^2 - [f^4(x) - f^4(u)]^2 = 0. \end{aligned} \tag{19}$$

Substituting (18) into (19), we obtain

$$\begin{aligned} &\left\{-\left[2a_1^1 a_1^3 + (a_1^2)^2\right] \cdot X^2 + \frac{\varepsilon}{X}(a_1^3)^2 - (a_1^4)^2\right\}(x^1 - u^1)^2 + \\ &+ \left\{-2\left(a_1^2 a_2^2 + a_1^1 a_2^3 + a_1^3 a_2^1\right) \cdot X^2 + \frac{2\varepsilon}{X} a_1^3 a_2^3 - 2a_1^4 a_2^4\right\}(x^1 - u^1)(x^2 - u^2) + \\ &+ \left\{-2\left(a_3^2 a_1^2 + a_1^1 a_3^3 + a_3^1 a_1^3\right) \cdot X^2 + \frac{2\varepsilon}{X} a_3^3 a_1^3 - 2a_1^4 a_3^4\right\}(x^1 - u^1)(x^3 - u^3) + \\ &+ \left\{-2\left(a_4^1 a_1^3 + a_1^1 a_4^3 + a_4^2 a_1^2\right) \cdot X^2 + \frac{2\varepsilon}{X} a_1^3 a_4^3 - 2a_4^4 a_1^4\right\}(x^1 - u^1)(x^4 - u^4) + \\ &+ \left\{-\left[2a_2^1 a_2^3 + (a_2^2)^2\right] \cdot X^2 + \frac{\varepsilon}{X}(a_2^3)^2 - (a_2^4)^2\right\}(x^2 - u^2)^2 + \\ &+ \left\{-2\left(a_2^1 a_3^3 + a_3^1 a_2^3 + a_2^2 a_3^2\right) \cdot X^2 + \frac{2\varepsilon}{X} a_2^3 a_3^3 - 2a_2^4 a_3^4\right\}(x^2 - u^2)(x^3 - u^3) + \\ &+ \left\{-2\left(a_2^1 a_4^3 + a_4^1 a_2^3 + a_2^2 a_4^2\right) \cdot X^2 + \frac{2\varepsilon}{X} a_2^3 a_4^3 - 2a_2^4 a_4^4\right\}(x^2 - u^2)(x^4 - u^4) + \\ &+ \left\{-\left[2a_3^1 a_3^3 + (a_3^2)^2\right] \cdot X^2 + \frac{\varepsilon}{X}(a_3^3)^2 - (a_3^4)^2\right\}(x^3 - u^3)^2 + \\ &+ \left\{-2\left(a_3^1 a_4^3 + a_4^1 a_3^3 + a_3^2 a_4^2\right) \cdot X^2 + \frac{2\varepsilon}{X} a_3^3 a_4^3 - 2a_3^4 a_4^4\right\}(x^3 - u^3)(x^4 - u^4) + \\ &+ \left\{-\left[2a_4^1 a_4^3 + (a_4^2)^2\right] \cdot X^2 + \frac{\varepsilon}{X}(a_4^3)^2 - (a_4^4)^2\right\}(x^4 - u^4)^2 = 0, \end{aligned} \tag{20}$$

where

$$X = \exp\left[\omega \sum_{k=1}^{4} a_k^4 u^k + \omega a^4\right]$$

If we now use Proposition 1.2 in Sec. 1, we can equate the coefficients of $(x^i - u^i)(x^k - u^k)$ in Eqs. (13) and (20). Equating the expressions multiplying $(x^1 - u^1)^2$ and $(x^1 - u^1)(x^2 - u^2)$, we obtain

$$a_1^4 = a_1^3 = a_1^2 = 0, \quad a_1^1 a_2^3 = 0. \tag{21}$$

Comparison of the coefficients multiplying $(x^4 - u^4)^2$ then gives

$$(a_4^4)^2 = 1, \quad a_4^3 = a_4^2 = 0;$$

for $(x^1 - u^1)(x^3 - u^3)$:

$$a_1^1 a_3^3 X^2 = e^{2\omega u^4}$$

or

$$a_1^4 = a_2^4 = a_3^4 = 0, \quad a_4^4 = 1, \quad a_1^1 a_3^3 = e^{-2\omega a^4}.$$

But then (21) implies $a_2^3 = 0$. Continuing with similar comparisons, we obtain the result

$$(a_2^2)^2 = e^{-2\omega a^4},$$

$$a_2^2 a_3^2 + a_2^1 a_3^3 = 0, \tag{22}$$

$$(a_3^2)^2 + 2a_3^1 a_3^3 = 0, \tag{23}$$

$$(a_3^3)^2 = e^{\omega a^4},$$

$$a_4^1 = 0.$$

Consequently, only the coefficients a_1^1, a_2^2, a_3^3, a_4^4, a_2^1, a_3^1, a_3^2 are nonzero. Moreover, the last three are related by (22) and (23). Hence one of them can be taken as an arbitrary parameter. Set $a_3^2 = \alpha_4 e^{2\alpha_5}$ and $a^4 = (-2/\omega)\alpha_5$. Then we have finally

$$\begin{aligned} &a_2^2 = \pm e^{2\alpha_5}; \quad a_3^3 = (\pm) e^{-\alpha_5}; \quad a_1^1 = (\pm) e^{5\alpha_5}; \quad a_4^4 = 1; \\ &a_3^2 = \alpha_4 e^{2\alpha_5}; \quad a_3^1 = -(\pm)\frac{\alpha_4^2}{2} e^{5\alpha_5}; \quad -a_2^1 = \pm(\pm)\alpha_4 e^{5\alpha_5}; \\ &a^1 = \alpha_1; \quad a^2 = \alpha_2; \quad a^3 = \alpha_3; \quad a^4 = -\frac{2}{\omega}\alpha_5, \end{aligned} \tag{24}$$

where the signs $\pm$ or $(\pm)$ appear where we had to take square roots, the parentheses in the second case appearing as a reminder that the choice of sign (say of a_3^3) automatically entails the choice of the same sign of a_1^1 and a_2^1, whereas there are no restrictions on the signs of a_2^2 and a_3^3.

If we now compare the affine map (18), (24) with the motion (5), we see easily that they coincide if the upper signs are chosen in (24). In general, the signs in (24) reflect the existence of two motions in $\overset{*}{T}_{2,3}$ given by

$$\text{(I)}\ \bar{x}^1 = -x^1,\ \bar{x}^3 = -x^3,\ \bar{x}^2 = x^2,\ \bar{x}^4 = x^4;$$

$$\text{(II)}\ \bar{x}^1 = x^1,\ \bar{x}^2 = -x^2,\ \bar{x}^3 = x^3,\ \bar{x}^4 = x^4,$$

which were not accounted for when we write (5). Thus case A) is proved.

B) Consider the space $\overset{*}{T}_{3,4}$.

Repeating all the arguments in part A) [not even changing the notation, apart from replacing expression (4) by (6)], we verify that f is affine in the coordinates which we consider, i.e., as before, Eq. (18) holds for f. It remains to calculate the coefficients a_k^i and a^i. We have two expressions, analogous to (13) and (19):

$$2\varepsilon_1 e^{-2\omega u^4}(x^1 - u^1)(x^3 - u^3) - e^{-2\omega u^4}(x^2 - u^2)^2 + 2\varepsilon_2 e^{\omega u^4}(x^2 - u^2)(x^3 - u^3) - \frac{1}{2}e^{4\omega u^4}(x^3 - u^3)^2 - (x^4 - u^4)^2 = 0 \tag{25}$$

$$\begin{aligned} &2\varepsilon_1 e^{-2\omega f^4(u)} \cdot [f^1(x) - f^1(u)][f^3(x) - f^3(u)] - e^{-2\omega f^4(u)} \cdot [f^2(x) - f^2(u)]^2 + \\ &+ 2\varepsilon_2 e^{\omega f^4(u)} \cdot [f^2(x) - f^2(u)][f^3(x) - f^3(u)] - \frac{1}{2}e^{4\omega f^4(u)} \cdot [f^3(x) - f^3(u)]^2 - [f^4(x) - f^4(u)]^2 = 0. \end{aligned} \tag{26}$$

Substituting (18) into (26), we obtain:

$$
\begin{aligned}
&\left\{\left[2\varepsilon_1 a_1^1 a_1^3 - (a_1^2)^2\right]\cdot X^{-2} + 2\varepsilon_2 X a_1^2 a_1^3 - \frac{X^4}{2}(a_1^3)^2 - (a_1^4)^2\right\}(x^1-u^1)^2 + \{2\left[\varepsilon_1\left(a_1^1 a_2^3 + a_2^1 a_1^3\right) - a_1^2 a_2^2\right]\cdot X^{-2} + 2\varepsilon_2 X\left(a_1^2 a_2^3 + a_2^2 a_1^3\right) - \\
&\quad - X^4 a_1^3 a_2^3 - 2a_1^4 a_2^4\}(x^1-u^1)(x^2-u^2) + \{2\left[\varepsilon_1\left(a_1^1 a_3^3 + a_3^1 a_1^3\right) - a_1^2 a_3^2\right]\cdot X^{-2} + \\
&\quad + 2\varepsilon_2 X\left(a_1^2 a_3^3 + a_3^2 a_1^3\right) - X^4 a_1^3 a_3^3 - 2a_1^4 a_3^4\}(x^1-u^1)(x^3-u^3) + \\
&\quad + \{2\left[\varepsilon_1\left(a_1^1 a_4^3 + a_4^1 a_1^3\right) - a_1^2 a_4^2\right)\cdot X^{-2} + 2\varepsilon_2 X\left(a_1^2 a_4^3 + a_4^2 a_1^3\right) - X^4 a_1^3 a_4^3 - 2a_1^4 a_4^4\}(x^1-u^1)(x^4-u^4) + \\
&\quad + \left\{\left[2\varepsilon_1 a_2^1 a_2^3 - (a_2^2)^2\right]\cdot X^{-2} + 2\varepsilon_2 X a_2^2 a_2^3 - \frac{X^4}{2}(a_2^3)^2 - (a_2^4)^2\right\}(x^2-u^2)^2 + \\
&\quad + \{2\left[\varepsilon_1\left(a_2^1 a_3^3 + a_3^1 a_2^3\right) - a_2^2 a_3^2\right]\cdot X^{-2} + 2\varepsilon_2 X\left(a_2^2 a_3^3 + a_3^2 a_2^3\right) - \\
&\quad - X^4 a_2^3 a_3^3 - 2a_2^4 a_3^4\}(x^2-u^2)(x^3-u^3) + \{2\left[\varepsilon_1\left(a_2^1 a_4^3 + a_2^3 a_4^1\right) - a_2^2 a_4^2\right]\cdot X^{-2} + \\
&\quad + 2\varepsilon_2 X\left(a_2^2 a_4^3 + a_4^2 a_2^3\right) - X^4 a_2^3 a_4^3 - 2a_2^4 a_4^4\}(x^2-u^2)(x^4-u^4) + \\
&\quad + \left\{\left[2\varepsilon_1 a_3^1 a_3^3 - (a_3^2)^2\right]\cdot X^{-2} + 2\varepsilon_2 X a_3^3 a_3^2 - \frac{X^4}{2}(a_3^3)^2 - (a_3^4)^2\right\}(x^3-u^3)^2 + \\
&\quad + \{2\left[\varepsilon_1\left(a_4^1 a_3^3 + a_4^3 a_3^1\right) - a_3^2 a_4^2\right]\cdot X^{-2} + 2\varepsilon_2 X\left(a_3^2 a_4^3 + a_4^2 a_3^3\right) - \\
&\quad - X^4 a_3^3 a_4^3 - 2a_4^4 a_3^4\}(x^3-u^3)(x^4-u^4) + \left\{\left[2\varepsilon_1 a_4^1 a_4^3 - (a_4^2)^2\right]\cdot X^{-2} + 2\varepsilon_2 X a_4^2 a_4^3 - \frac{X^4}{2}(a_4^3)^2 - (a_4^4)^2\right\}(x^4-u^4)^2 = 0,
\end{aligned}
\tag{27}
$$

where

$$X = \exp\left[\omega\sum_{k=1}^{4} a_k^4 u^k + \omega a^4\right]$$

We now compare (25) with (27). The expressions multiplying $(x^1 - u^1)^2$ and $(x^4 - u^4)^2$ give

$$a_1^2 = a_1^3 = a_1^4 = 0, \quad a_4^2 = a_4^3 = 0, \quad (a_4^4)^2 = 1,$$

and further comparison of the coefficients of $(x^1 - u^1)(x^3 - u^3)$, $(x^2 - u^2)^2$, $(x^3 - u^3)^2$ gives:

$$a_1^1 a_3^3 X^{-2} = e^{-2\omega u^4}, \quad \text{i.e.,} \quad a_2^4 = a_3^4 = 0, \quad a_4^4 = 1, \quad a_1^1 a_3^3 = e^{2\omega a^4}, \tag{28}$$

$$a_2^3 = 0, \quad a_2^2 = \pm e^{\omega a^4}, \quad a_3^1 = a_3^2 = 0, \quad a_3^3 = \pm e^{-2\omega a^4}. \tag{29}$$

while the coefficients of $(x^2 - u^2)(x^3 - u^3)$ and $(x^3 - u^3)(x^4 - u^4)$ give

$$a_2^1 = 0, \quad a_2^2 a_3^3 = e^{-\omega a^4}, \quad a_4^1 = 0. \tag{30}$$

Equations (28) and (29) imply that $a_1^1 = \pm e^{4\omega a^4}$.

Thus the only nonzero coefficients are a_1^1, a_2^2, a_3^3, a_4^4, and as we see from (28), (30), the signs of a_1^1, a_2^2, a_3^3 are all the same. Finally, we have

$$
\begin{aligned}
\bar{x}^1 &= (\pm)\, e^{4\omega a^4} x^1 + a^1, \\
\bar{x}^2 &= (\pm)\, e^{\omega a^4} x^2 + a^2, \\
\bar{x}^3 &= (\pm)\, e^{-2\omega a^4} x^3 + a^3, \\
\bar{x}^4 &= x^4 + a^4.
\end{aligned}
\tag{31}
$$

Letting $a^4 = \omega^{-1}\beta$, $a^3 = \delta$, $a^2 = \gamma$, $a^1 = \alpha$, then we see easily upon comparing the affine transformation (31) with the motion (7) that they coincide if the upper signs are chosen in (31).

The choice of lower signs indicates the existence of a transformation of the form $\bar{x}^1 = -x^1$, $\bar{x}^2 = -x^2$, $\bar{x}^3 = -x^3$, which is easily seen to be a motion in $\overset{*}{T}_{3,4}$ which was not accounted for when we wrote down (7).

Thus, f is a motion. Hence the assertion of the theorem is also true for the space $\overset{*}{T}_{3,4}$.

The theorem is proved.

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STATISTICS OF THE SPECTRAL DENSITIES OF STATIONARY STOCHASTIC PROCESSES

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In this article we are interested in the construction and investigation of the asymptotic properties of the statistics of the spectral density $f(\lambda)$ that are constructed with respect to a sample $\{X(1), \ldots, X(N)\}$ of a stationary stochastic process $X(t)$, $t = \ldots -1, 0, 1, \ldots$, with the mean $MX(t) = 0$ and the covariance function $C(t)$. It is natural to consider the class of all the quadratic forms

$$\frac{1}{2\pi N} \sum_{s,t=1}^{N} b_{s,t}^{(N)} X(s) X(t) \tag{0.1}$$

with arbitrary coefficients $b_{s,t}^{(N)}$ as the admissible class of the statistics for reasons of dimension. As shown by Grenander and Rosenblatt [1-3], the asymptotic behavior for $N \to \infty$ of the first two moments of the statistic of the spectral density is not worsened if in place of the class (0.1) of statistics we consider the narrower class of the statistics of the form

$$\hat{f}_N(\lambda) = \frac{1}{2\pi M} \sum_{s,t=1} b^{(N)}(t-s) e^{i(t-s)\lambda} X(t) X(s), \tag{0.2}$$

which can be represented in the form

$$\hat{f}_N(\lambda) = \frac{1}{2\pi} \sum_{t=-N+1}^{N-1} e^{it\lambda} b^{(N)}(t) C_N(t), \tag{0.3}$$

where

$$C_N(t) = \frac{1}{N} \sum_{s=1}^{N-|t|} X(s) X(s+|t|). \tag{0.4}$$

Under quite general conditions they have shown that for arbitrary asymptotically unbiased estimate $\hat{f}_N^*(\lambda)$ of the class (0.1) there exists a statistic $\hat{f}_N(\lambda)$ of the class (0.2) such that its bias

$$\Delta\hat{f}_N(\lambda) = M\hat{f}_N(\lambda) - f(\lambda) \tag{0.5}$$

coincides with the bias of the statistic $\hat{f}_N^*(\lambda)$, and the variance is asymptotically less than or equal to the variance of $\hat{f}_N^*(\lambda)$. Thus, from the point of view of asymptotic behavior, it is sufficient to consider statistics of the class (0.2).

The statistic $\hat{f}_N(\lambda)$ can also be represented in the form

$$\hat{f}_N(\lambda) = \int_{-\pi}^{\pi} \Phi_N(X) I_N(x+\lambda)\, dx, \tag{0.6}$$

where $I_N(x)$ is called the periodogram

$$I_N(x) = \frac{1}{2\pi N} \left| \sum_{t=1}^{N} e^{itx} X(t) \right|^2 \tag{0.7}$$

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