AUTOMORPHISMS OF LOCAL ORDER STRUCTURES ON HYPERBOLIC MANIFOLDS

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In [1-3], groups of order automorphisms in Euclidean and hyperbolic spaces were calculated. In [4] local order automorphisms generated by elliptic cones in Euclidean and conformal spaces were described. The present article shows how, by methods developed in [4], to calculate automorphisms of local (elliptic) orders on hyperbolic manifolds.

1. Order Structures and Local Order Structures. We shall call the family $\mathscr{P}=\{P_x:\ x\in V\}$ of subsets of a set V an order structure if the following conditions are satisfied: 1) $x\in P_x$, 2) implies $P_y \subset P_x$, 3) if $x \neq y$ then $P_x \neq P_y$.

Let V be a topological space and $\mathcal{O}=\{O_x:\ x\in V\}$ be a covering of V by open neighborhoods O_x of points $x\in V$. We assume also that on each O_x, an order structure $\mathscr{P}_x=\{P_{xa}:\ a\in O_x\}$ is assigned. By $\mathscr{P}_x|U$, where U is a set in V, we denote the family $\{P_{xa}\cap U:\ a\in U\}$ and call it the boundary of order $\mathscr{P}_x$ on set U.

A local order structure on V is the family $\mathscr{P}=\{\mathscr{P}_x:\ x\in V\}$ of order structures of covering , where the family satisfies condition $\mathscr{P}_x|O_x\cap O_y=\mathscr{P}_y|O_x\cap O_y$ for any intersecting neighborhoods O_x, O_y.

It is obvious that if $\mathscr{P}$ is an order structure on V, then $\{\mathscr{P}|O_x:\ x\in V\}$ is a local order structure on V. Thus, a "global" order structure always defines a local one. The converse statement is false in general.

Definition 1. Let $\mathscr{P}$ be a local order structure on V and let f: V → V be a bijection. The bijection is called a local order automorphism or simply a $\mathscr{P}$-automorphism, if for each point $x\in V$ there exists a connected open neighborhood $W_x\subset O_x$ such that $f(W_x)\subset O_{f(x)}$ and f $(\mathscr{P}_x|W_x)=\mathscr{P}_{f(x)}|f(W_x)$, i.e., $f(P_{xa}\cap W_x)=P_{f(x)f(a)}\cap f(W_x)$ for each point $a\in W_x$.

If U ⊂ V is a domain, and f: U → V is an injection, then f is called a $\mathscr{P}$-monomorphism on U, if for any $x\in U$ there exists a connected open neighborhood $W_x\subset O_x\cap U$ such that $f(W_x)\subset O_{f(x)}$ and $f(P_{xa}\cap W_x)=P_{f(x)f(a)}\cap f(W_x)$ for any point $a\in W_x$.

2. Local Order on Hyperbolic Manifolds. We shall call complete connected Riemannian manifolds of constant curvature −1 hyperbolic manifolds. It is known that hyperbolic manifolds are exactly the factor-spaces H^n/Γ of hyperbolic space H^n over freely operating discrete group Γ of its motions [5, Corollary 2.4.10]. Moreover, hyperbolic manifold M^n is locally isometric to space H^n [5, Corollary 2.3.17, Corollary 2.3.8].

Let $p\colon H^n\to H^n/\Gamma\equiv M^n$ be the natural projection. Since Γ acts on H^n completely discontinuously, for each point $x\in H^n$ we can choose an open neighborhood U_x such that for any action $\gamma\in\Gamma$, the neighborhood $U_{\gamma(x)}\equiv\gamma(U_x)$ is isometric to $p(U_x)\equiv O_{p(x)}$, i.e., a neighborhood of point $p(x)\in M^n$. In other words, for each point $z\in M^n$ we can choose an open neighborhood O_z such that $p^{-1}(O_z)$ will be the union of nonintersecting isometric open neighborhoods U_x of points x, where the U_x belong to orbit $p^{-1}(z)$. We denote the family of such neighborhoods by $\mathcal{O}=\{O_z:\ z\in M^n\}$.

Let $\mathscr{P}=\{P_x:\ x\in H^n\}$ be a hyperbolic order structure on H^n, i.e., $\mathscr{P}$ is the order structure on H^n such that for some simply transitive subgroup T of the group of actions of space H^n, the equation $t(P_x)=P_{t(x)}$ is satisfied for any $x\in H^n$ and $t\in T$ [2, 3].

Further, we shall assume that $\mathscr{P}$ is invariant with respect to the section of the discrete group of motions Γ (for example, Γ ⊂ T), i.e., $\gamma(P_x)=P_{\gamma(x)}$ for $\gamma\in\Gamma$.

Consider family $\Pi=\{\mathscr{P}_z:\ z\in M^n\}$, where $\mathscr{P}_z=\{p(P_a\cap U_{x(z)}):\ a\in U_{x(z)}\}$ for some point $x(z)\in p^{-1}(z)$. From the definition of family $\mathcal{O}$, it is obvious that the definition of $\mathscr{P}_z$ does not depend on the choice of point $x(z)\in p^{-1}(z)$ because all neighborhoods U_x of points of orbit $p^{-1}(z)$ are isometric, and the corresponding isometries belong to Γ.

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Assume that $O_{z_1} \cap O_{z_2} \neq \varnothing$, $z_1, z_2 \in M^n$, and $z \in O_{z_1} \cap O_{z_2}$. Then take $u, v \in p^{-1}(z)$ such that $u \in U_{x_1}$, $v \in U_{x_2}$, where $x_i \in p^{-1}(z_i)$ (i = 1, 2). If $U_{x_1} \cap U_{x_2} \neq \varnothing$, then $O_{z_1} \cap O_{z_2} = p(U_{x_1} \cap U_{x_2})$ and therefore

$$\mathscr{P}_{z_1} | O_{z_1} \cap O_{z_2} = p(\mathscr{P} | U_{x_1} \cap x_2) = \mathscr{P}_{z_2} | O_{z_1} \cap O_{z_2}. \tag{1}$$

Let us assume that $U_{x_1} \cap U_{x_2} = \varnothing$. Since u, v lie in the same orbit, there exists $\gamma \in \Gamma$ such that v = γ(u). But then $U_{x_2} \cap \gamma(U_{x_1}) \neq \varnothing$ and since $\gamma(U_{x_1}) = U_a$ for some point $a \in p^{-1}(z_1)$ then $U_{x_2} \cap U_a \neq \varnothing$. Consequently, up to redenotation $a \to x_1$, the condition $U_{x_1} \cap U_{x_2} \neq \varnothing$ is satisfied, i.e., Eq. (1) is valid. This indicates that Π is a local order structure on M^n, which we shall call a locally hyperbolic order structure on a hyperbolic manifold.

Let f: U → M^n, where U ⊂ M^n is an open set, Π a monomorphism. If $x \in U$, then take continuous path $\omega \subset M^n$ connecting x with f(x). For each point $u \in p^{-1}(x)$ there exists a lifting $\tilde{\omega}$ of path ω with initial point u. Define mapping $\tilde{f}\colon p^{-1}(U) \to H^n$ assuming that $\tilde{f}(u)$ is the final point of path $\tilde{\omega}$, i.e., $p \circ \tilde{f}(u) = f(x)$. Our definition of $\tilde{f}$ is correct because two liftings of a path with different initial points have distinct final points [5, Theorem 1.8.3]. Consequently, $\tilde{f}$ is an injection which, as it is easy to see, is a local order monomorphism with respect to local order structures $\tilde{\Pi}$, where $\tilde{\Pi} = \{\tilde{\mathscr{P}}_x : x \in H^n\}$, $\mathscr{P}_x = \{P_a \cap U_x : a \in U_x\}$. From this it follows that properties of a Π-monomorphism f: U → M^n are defined by properties of its lifting $\tilde{f}$: $p^{-1}(U) \to H^n$ and therefore the calculation of Π-monomorphisms reduces to the calculation of $\tilde{\Pi}$-monomorphisms.

In this article we will investigate local order monomorphisms with respect to elliptic local orders on M^n, defined by circular quasicones [3] on H^n, n ≥ 3, i.e., in H^n we consider the hyperbolic order structure $\mathscr{P}$, where in Poincaré's model P_x is represented by a circular cone. We shall call the corresponding local order structure Π on M^n an elliptic locally hyperbolic order structure. For the Poincaré model, we regard the Lobachevskii space H^n to be semispace $\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 > 0\}$, where $\mathbf{R}^n$ is the n-dimensional arithmetic space with metric

$$ds^2 = \frac{1}{x_1^2} \sum_{i=1}^{n} dx_i^2.$$

We denote the representation in the Poincaré model of an object A of hyperbolic geometry by |A|.

In space $\mathbf{R}^n$ we introduce a scalar product

$$x \cdot y = x_1 \cdot y_1 - \sum_{i=2}^{n} x_i \cdot y_i, \quad x^2 = x \cdot x. \tag{2}$$

Space $\mathbf{R}^n$ provided with scalar product (2) is equipped in the standard way with the structure of a pseudo-Euclidean space. Further, by conformal mappings we have in mind conformal mappings of the indicated pseudo-Euclidean space. They can be represented in the form of a superposition of homotheties $x \to \lambda x + a$, $\lambda \neq 0$, Lorentz transformations, i.e., affine bijections preserving x^2, and inversions $x \to \frac{x-a}{(x-a)^2} + a$ [4].

<u>Definition 2.</u> An injective mapping $g\colon U \to H^n$ ($U \subset H^n$ is a domain) is called locally q-conformal if for each point $x \in U$ there exist a neighborhood $W_x \subset U$ and a conformal mapping g_x such that $|g|W_x| = g_x$. The injection $g\colon U \to M^n$ ($U \subset M^n$ is a domain) is locally q-conformal if its lifting $\tilde{g}\colon p^{-1}(U) \to H^n$ is locally q-conformal.

<u>THEOREM.</u> Let Π be an elliptical locally hyperbolic order structure on hyperbolic manifold M^n, n ≥ 3. Then any Π-monomorphism $f\colon U \to M^n$ ($U \subset M^n$ is a domain) is locally q-conformal.

<u>Proof.</u> It is sufficient to prove the assertion of the theorem for lifting $\tilde{f}\colon p^{-1}(U) \to H^n$ which is a $\tilde{\Pi}$-monomorphism. According to Definition 1, there exists a connected open neighborhood W_x for each point $x \in p^{-1}(U)$, $W_x \subset p^{-1}(U)$ for which

$$\tilde{f}(P_a \cap W_x) = P_{\tilde{f}(a)} \cap \tilde{f}(W_x)$$

or

$$|\tilde{f}|(|P_a| \cap W_x) = |P_{\tilde{f}(a)}| \cap |\tilde{f}|(W_x) \tag{3}$$

for any point $a \in W_x$. From (3) it follows (by a theorem of Aleksandrov [4, p. 8]) that $|\tilde{f}|$ is conformal on W_x, i.e., $\tilde{f}$ is locally q-conformal. The theorem is proved.

Remark. The Euclidean analogue of the theorem follows from results obtained by Aleksandrov [4] and by Lester [6]. The corresponding statement is obtained from the statement of the theorem if the words "hyperbolic" and "q-conformal" are changed, respectively, to "Euclidean" and "conformal." In this case the words "Euclidean manifold M^n" mean a flat manifold $\mathbf{R}^n/\Gamma$.

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NON-ABELIAN FREE PRO-p-GROUPS CANNOT BE REPRESENTED BY 2-BY-2 MATRICES

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It is well known [1] that a free discrete group admits a faithful representation of degree 2 over the ring Z. When looking at topological groups, one naturally tries to find continuous representations over commutative topological rings. Our main result is the following: for p ≠ 2 the free non-Abelian pro-p-group does not admit a faithful matrix representation of degree 2 over any associative and commutative profinite ring with identity. It follows, in particular, that for p ≠ 2 every pro-p-group in $GL_2(K)$ (K is an arbitrary commutative and associative profinite ring with identity) satisfies a standard "topological" identity that is independent of K.

1. Definitions. Necessary Information and Results

We will assume that all rings are associative and have an identity. An inverse limit of finite rings (resp. groups) is called a profinite ring (resp. profinite group). All subrings and subgroups are closed. Quotient rings and quotient groups are endowed with the quotient topologies. Homomorphisms are continuous.

Let F(X) be the free discrete group on the set X. The free pro-p-group $F_p(X)$ is the completion of F(X) in the topology defined by the family of all normal subgroups N_i, $i \in I$, of index a power of the prime p, that contain almost all the generators in X. We list some of the properties of $F_p(X)$:

a) the subgroups of $F_p(X)$ are free pro-p-groups (the analog of Schreier's theorem);

b) if $|X| \geqslant 2$, then the nonidentity normal subgroups of $F_p(X)$ are non-Abelian;

c) if $|X| = n$, $n < \infty$, then the r-th quotient of the lower central series is a free Z_p-module of rank

$$l_n(r) = \frac{1}{r} \sum_{m/r} \mu(m)\, n^{r/m}.$$

Let K be a finite commutative ring, then the set $K_p = \{x \in K \mid (\exists n)\, p^n x = 0\}$ is an ideal in K. Moreover, $K = \bigoplus_p K_p$ for every prime p. Suppose now that K is a commutative profinite ring,

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