of (pairwise nonisomorphic) finitely generated indecomposable right R-modules. An arbitrary right R-module over such a ring is a direct sum of finitely generated modules [11].

COROLLARY 5. Let R be a ring of finite representation type and $\mathscr{M}$ = Mod-R be the category of right R-modules. Then Eq. (1) is valid for each inductively closed proper class $\mathscr{P}$ in $\mathscr{M}$.

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# DENSE ORDER IN LOBACHEVSKII SPACE


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Let $H^n$ be an n-dimensional hyperbolic space, let T be a simply transitive subgroup of the group of motions, whose Lie algebra is isomorphic to the Lie algebra of the group of affine transformations of the arithmetic space $R^{n-1}$ generated by translations and homotheties, and let $\mathscr{P} = \{P_x\colon\ x \in H^n\}$ be a preorder in $H^n$, that is, a family of subsets of $H^n$ that satisfies the conditions: 1) $x \in P_x$, 2) if $y \in P_x$, then $P_y \subset P_x$, 3) for any $x \in H^n$ and $t \in T$ we have $t(P_x) = P_{t(x)}$.

We put $P_x^- = \{y \in H^n\colon\ x \in P_y\}$.

Definition. A preorder $\mathscr{P}$ is said to be dense if for any x, y such that $y \in P_x \setminus \{x\}$, we have $P_x \cap P_y^- \neq \{x, y\}$.

If the relation $y \in P_x$ is written as $x \leqslant y$, that is, we use the traditional notation for the preorder, then density of the preorder means that $x \leqslant y$ implies the existence of $z \neq x, y$ such that $x \leqslant z$ and $z \leqslant y$, where it is assumed that $x \neq y$.

A preorder $\mathscr{P}$ is closed if every set $P_x$ is closed.

The aim of this note is to describe closed dense preorders in $H^n$, $n \geq 2$. For affine space $A^n$, $n \geq 2$, this problem was posed by A. D. Aleksandrov and solved by A. V. Levichev [1].

1. A quasiline (quasiray with origin x) passing through a point x is the orbit of x under a one-parameter subgroup (semigroup) of the group T. Correspondingly an m-dimensional


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quasiplane passing through x is the orbit of x under an m-dimensional subgroup of T. Quasilines (q-lines) and also quasiplanes (q-planes) have many of the properties of lines and planes respectively in affine space (for the details see [2]).

A quasiray (q-ray) with origin x passing through y, $y \neq x$, is denoted below by $L(x, y)$. Also, as a rule, instead of the word "quasihyperplane" we write "quasiplane."

A quasicone (q-cone) $C_x$ with vertex x is a union of q-rays with origin at x.

Let $\mathscr{P}$ be a preorder in $H^n$ and e a fixed point in $H^n$. We put $P \equiv P_e$. We denote by $\overline{A}$ the closure of a set A, and by $A_x$ the image of A under the action of a motion $t \in T$ such that $t(e) = x$, that is, $A_x = t(A)$.

The quasicontingency (q-contingency) of a set P at the point e is the q-cone formed by all possible limits of q-rays emanating from e and passing through $x \in P$, $x \neq e$, as x tends to e. If $e \notin \overline{P\setminus\{e\}}$, we assume that the q-contingency of P at e is the point {e}. We denote the quasicontingency of P at e by $qc(P, e)$.

Proposition 1. $qc(P, e) \subset \overline{P}$.

For the proof see [2].

Proposition 2. Let $\mathscr{P}$ be a dense preorder in $H^n$ such that $P \neq \{e\}$ and for any $x \in P$ the set $\overline{P \cap P_x^-}$ is compact. Then $e \in \overline{P\setminus\{e\}}$, and so $qc(P, e) \neq \{e\}$.

Proof. Suppose that $x \in P$, that is, $e \leq x$. Then there is an $x_1$ such that $e \leq x_1 \leq x$, and so on. As a result we construct a sequence $\{x_n\} \subset P \cap P_x^-$ for which $e \leq x_{n+1} \leq x_n$ for every n. Since $\overline{P \cap P_x^-}$ is compact, without loss of generality we may assume that $\{x_n\}$ converges to a point $a$. Let U be a neighborhood of $a$. Starting from some number, all $x_n \in U$. We take two points $x_n, x_m \in U$, where $x_n \leq x_m$. Let $t \in T$ be a motion such that $t(x_n) = e$. Then $t(x_m) \in t(U)$ and $e \leq t(x_m)$. Because the neighborhood U is arbitrary, this implies that $e \in \overline{P\setminus\{e\}}$. This proves Proposition 2.

2. The exterior q-cone is the set $\exp P = \overline{\bigcup_{x\in P, x\neq e} L(e, x)}$.

THEOREM. Let $\mathscr{P}$ be a dense closed preorder in $H^n$, $n \geq 2$, such that the exterior q-cone ext P does not contain a q-line. Then P is a q-cone, more precisely, $P = qc(P, e)$.

Proof. Since ext P does not contain a q-line, there is a strictly supporting quasiplane E to ext P at e, that is, $\text{ext } P \cap E = \{e\}$, and therefore $P \cap E = \{e\}$. The quasiplane E splits $H^n$ into two connected components. Let $E^+$ be the open connected component containing $P\setminus\{e\}$, and $E^- = H^n\setminus\overline{E^+}$. Suppose the contrary, that is, P is not a quasicone. Then there is a point $x_1 \in E^+ \cap (P\setminus C)$, where $C = qc(P, e)$. Let $Q_1$ be the strictly supporting quasiplane to C at e that passes through $x_1$, and let $Q_1^+$ be the open connected component of the set $H^n\setminus Q_1$ that contains C, $Q_1^- = H^n/\overline{Q_1^+}$.

By hypothesis $P \cap P_{x_1}^- \neq \{e, x_1\}$, so there is a point $a \in P \cap P_{x_1}^-$ such that $a \neq e, x_1$.

Three cases are possible:

$$1)\ a \in Q_1^- \cap E^+ \cap E_{x_1}^-,$$

$$2)\ a \in Q_1 \cap E^+ \cap E_{x_1}^-,$$

$$3)\ a \in Q_1^+ \cap E^+ \cap E_{x_1}^-.$$

In Case 1 we take $x_2 = a$ and for $x_2$ we repeat the arguments for $x_1$, that is, we take the strictly supporting q-plane $Q_2$ to C that passes through e and $x_2$, and then choose a point of $P \cap P_{x_2}^-/(E \cup E_{x_2})$, for which it is again necessary to analyze the three cases mentioned above (we note that to complete the following inductive step there is no need to require that the condition $x_{n+1} \in P \cap P_{x_n}^-$ is satisfied – see the analysis of Case 3 below: every subsequent step is possible if at least one of the conditions of the form 1-3 is satisfied).

In case 3 let $t \in T$ be a motion that takes $a$ to $x_1$. Then $t(x_1) \in P_{x_1}$ and $t(x_1) \in Q_1^-$. If we take $\tau \in T$ so that $\tau(x_1) = e$, then $\tau t(x_1) \in P \cap Q_1^-$, since $\tau(Q_1) = Q_1$. Moreover, since $a \in E^+ \cap E_{x_1}^-$ we have $t(x_1) \in E_{x_1}^+ \cap E_{t(x_1)}^-$, and so $\tau t(x_1) \in E^+ \cap E_{x_1}^-$. Thus, $\tau t(x_1) \in P \cap Q_1^- \cap E^+ \cap E_{x_1}^-$. Thus from Case 3 we have arrived at Case 1.

In Case 2 we can repeat the arguments carried out for the point $x_1$. Again we arrive at Cases 1-3. If each time we just repeat Case 2, then we obtain the sequence $\{x_n\} \subset Q_1 \cap E^+ \cap E_{x_1}^-$, where $x_n \in P \cap Q_1 \cap E^+ \cap E_{x_{n-1}}^-$. Since the set $P \cap E^+ \cap E_{x_1}^-$ is compact, without loss of generality we may assume that $\{x_n\}$ converges to a point $b \in P \cap Q_1 \cap E^+ \cap E_b^-$. Since $E_b \subset E_{x_1}^-$, we have succeeded in going from $x_1$ to a point b "closer" to e than $x_1$, in the sense that $E_b$ lies between E and $E_{x_1}$. Now, redenoting b by $x_1$, we repeat all the arguments afresh. If we again have only Case 2, then we obtain a point b' still "closer" to e than b was before, that is, $E_{b_1}$ lies between E and $E_b$. In transfinitely many steps we arrive at the point e, that is, we distinguish a sequence of points $\{b_n\} \subset P \cap Q_1 \cap E^+ \cap E_{x_1}^-$ that converges to e. Then the q-rays $\{L(e, b_n)\}$ will have as limit a q-ray L lying in $Q_1$. But by definition $L \subset C$, and by construction $L \cap qc(P, e) = \{e\}$. This is a contradiction.

However, we now turn to Case 1 and Case 3 which reduces to it.

In Case 1, repeating our arguments time after time, we arrive at a sequence of points $x_n \in P \cap Q_{n-1}^- \cap E^+ \cap E_{x_{n-1}}^-$, where $Q_n$ is the strictly supporting quasiplane to C passing through e and $x_n$, and all the $Q_n$ intersect in one $(n-2)$-dimensional q-plane. Since $P \cap E^+ \cap E_{x_1}^-$ is compact, without loss of generality we may assume that $\{x_n\}$ converges, moreover to the point e (the last part is obtained just as in the analysis of Case 2). But then the q-rays $\{L(e, x_n)\}$ have as limiting q-ray $L \subset Q_1^-$. By the definition of the quasicontingency $L \subset qc(P, e) \equiv C$, and by construction $L \cap C = \{e\}$. This is a contradiction.

Thus the assumption that $P \backslash C \neq \emptyset$ leads to a contradiction. Therefore $P = C$. This proves the theorem.

3. A semigroup P of a Lie group is said to be Lie if it is the closure of the subsemigroup S generated by the union of all one-parameter semigroups that are contained in S (see [3]). Obviously an arbitrary closed subsemigroup need not be Lie. However, taking into account the importance [3] of the concept of a Lie subsemigroup, it is useful to be able to establish that subsemigroups of a Lie group are Lie.

We say that a subsemigroup P containing the identity is dense if for each $x \in P$ there is an $a \in P$ such that $a^{-1}x \in P$. Since a subsemigroup P containing the identity generates a left-invariant preorder $\mathscr{P} = \{P_x\colon\ P_x = x \cdot P,\ x \in G\}$ on the Lie group G, the given definition of a dense subsemigroup implies that the preorder $\mathscr{P}$ is dense.

The theorem proved in this note, together with a theorem of A. V. Levichev [1], gives sufficient conditions for a closed dense subsemigroup to be Lie in the case when the Lie group G is isomorphic either to the group of parallel displacements of Euclidean space $E^n$, $n \geq 2$, or to the simply transitive group of motions of Lobachevskii space $H^n$, $n \geq 2$ (the basic affine Lie group in the terminology of [4]). Both these groups are exceptional among ordered Lie groups [4]. Consequently, the question of whether closed dense subsemigroups of Lie groups are Lie has been solved for two exceptional groups and remains open for all the others.

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