CLOSED TIMELIKE SMOOTH CURVES IN THE GENERAL THEORY OF RELATIVITY

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For a space—time which admits a closed timelike smooth curve it is estimated that $\tau \sim 2 \cdot 10^{-24} \cdot \sqrt{\rho}\, l^2$, where τ is the real time and l the spatial length associated with the timelike curve, and ρ is the density of material.

In connection with Howard's paper [1] dealing with the cosmological model of Gödel [2] and particularly with Gödel's statement that a closed timelike smooth curve exists in his model, it is important to reconsider this interesting problem in the general theory of relativity. Howard casts some doubt on the result of Chandrasekhar and Wright [3] that a closed timelike smooth geodesic is impossible in the Gödel model. It is shown below that the original conclusion in [3] is correct.

Different opinions have been expressed about models which admit closed timelike smooth curves (timelike cycles)(see [5]; [8]; [6], p. 625. The estimates which we make below, however, show that the phenomena either cannot be observed in practice, or are only realized in areas where modern physics has not yet penetrated, or must be considered from, say, a quantum-mechanical rather than a classical point of view.

As regards the allegations that the principle of causality is infringed by models which admit timelike cycles, it is better to turn to philosophy ([7], Causality) and it then becomes clear that the fears are unfounded. Information on timelike cycles can be found in [9].

1. The Gödel metric has the form

$$ds^2 = a^2 \left(dx^{0^2} - dx^{1^2} + \frac{1}{2} e^{2x^1} dx^{2^2} - dx^{3^2} + 2e^{x^1} dx^0 dx^2\right), \tag{1}$$

where a = const, and the variables x^0, x^1, x^2, x^3 can have all numerical values.

We assume that the metric admits a timelike cycle and that this is given by the relationships

$$\left.\begin{aligned} x^i &= f_i(t), \quad t \in [0,1] \\ f_i(0) &= f_i(1), \ \frac{df_i}{dt}(0) = \frac{df_i}{dt}(1) \end{aligned}\right\} (i = 0, 1, 2, 3),$$

where f_i is a function of class C^k $(k \geq 1)$. It is not difficult to see that the function f_2 cannot be constant and this implies that the function $\xi(t) = f_0'/f_2'$, (where the dash denotes differentiation with respect to t) can vary over the whole numerical axis, because f_2 has an extremum in the interval (0, 1).

We thus have

$$(s')^2 = [f_0' + e^{f_1(t)} f_2']^2 - (f_1')^2 - \frac{e^{2f_1(t)}}{2}(f_2')^2 - (f_3')^2 \leqslant [f_0' + e^{f_1(t)} f_2']^2.$$

Let $\{I_i\}_{i=1}^{m}$ $(m \geq 1)$, the intervals containing the zeroes of the function $f_2'(t)$, be so small that

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$$A \equiv [0,1] \setminus \bigcup_{i=1}^{m} I_i \neq \varnothing.$$

Then for A we have: $(s')^2 \leq (f_2')^2 F(t)$, where $F(t) \equiv [\xi(t) + \exp f_1(t)]^2$. We take the manifold $\bigcup_{i=1}^{m} I_i$ such that at the ends of the intervals $I_i (i = 1, \ldots, m)$

$$|\xi(t)| > \exp f_1(t), \tag{2}$$

and also $A \neq \varnothing$. Suppose that $t_0 \in (0, 1)$ is the extremal point of the function f_2. Then $t_0 \in I_k$, and as a result of (2) and of the fact that $\xi(t)$ has a zero at t_1 in the interval (0, 1) (with of course $t_1 \neq t_0$) it immediately follows that the graphs of the functions $\xi(t)$ and $\exp f_1(t)$ have a common point, i.e., there exists a point t_2 such that $t_2 \in A$ and $\xi(t_2) = -\exp f_1(t_2)$. But then $(s')^2(t_2) \leq 0$, and this contradicts the timelike nature of our curve.

Thus the Gödel metric, considered as a cosmological model with a Euclidean topology, does not admit timelike cycles. The conclusion of Chandrasekhar and Wright is therefore correct.

2. In our opinion, Gödel's error occurred as a result of the careless use of a coordinate transformation, because a change of coordinates implies the possibility of a transition from one topology to another. We can make this clear by means of an example. From a Minkowski plane, i.e., a plane with the metric

$$ds^2 = dx^2 - dy^2, \tag{3}$$

we cut out the strip $\{0 \leq x \leq 1\}$ and identify the boundary points (0, y) with (1, y). We thus go over from a Euclidean to a cylindrical topology and thus ensure the existence of a timelike cycle which did not exist before. The procedure is also contained in the transformation

$$x = \frac{1}{2}(\xi^2 + \eta^2)\,\mathrm{sharctg}\,(\xi/\eta), \qquad y = \frac{1}{2}(\xi^2 + \eta^2)\,\mathrm{charctg}\,(\xi/\eta), \tag{4}$$

which converts (3) to the form

$$ds^2 = (\eta^2 - \xi^2)(d\xi^2 - d\eta^2) - 4\xi\eta\, d\xi\, d\eta, \tag{5}$$

because (4) contains the transformation exp (x + iy) which "picks out" a point from the plane. The timelike cycle for (5) is given by the relationships $\xi = \sin t$, $\eta = \cos t$ and the velocity of a particle from the given world line at the instant the cycle is closed is equal to $c \cdot \mathrm{th} 4\pi \approx 0.99c$!

3. We now derive some estimates which enable us to judge under what physical conditions timelike cycles are realized.

We use the notation and terminology of [4].

We suppose that the gravitational field is constant and is created by macrodust at rest. Since the real time and spatial distance are chronometrically invariant, we can take $g_{00} = \mathrm{const} > 0$, for in the opposite case it is possible to make the transformation $x^0 \to \sqrt{g_{00}}\, x^0$, $x^\alpha \to x^\alpha$.

We assume that the timelike cycle L is an analytical Jordan curve, that it lies on the surface $F\{x^0, x^3 = \mathrm{const}\}$ and bounds the region F. We can assume without loss of generality that the cycle L is given by the conditions ${x^1}^2 + {x^2}^2 = \mathrm{const}$; x^0, $x^3 = \mathrm{const}$, i.e., is a "circle," because if it is not we can make an analytical transformation of the coordinates x^1 and x^2 by virtue of the Riemann theorem and the Schwartz principle [10]. Suppose also that $g_{03} = g_{13} = g_{23} = 0$.

Calculating the real time spent round the path L, we get

$$\tau(L) = \frac{1}{c} \oint_L \sum_\alpha \frac{g_{0\alpha}}{\sqrt{g_{00}}}\, dx^\alpha = \iint_F \Omega\, dx^1 dx^2,$$

where

$$\Omega = \frac{\sqrt{g_{00}}}{c} \left\{ \frac{\partial}{\partial x^1}\left(\frac{g_{02}}{g_{00}}\right) - \frac{\partial}{\partial x^2}\left(\frac{g_{01}}{g_{00}}\right) \right\}.$$

On the basis of the equation in [4], we can write

$$\Omega^2 = \frac{8\pi G\rho}{c^4} \frac{1}{g^{11}g^{22} - (g^{12})^2},$$

where ρ is the density of material. Noting that the tensor $\gamma_{\alpha\beta}$ gives a cylindrical Riemannian metric on the surface F and denoting the determinant of the matrix $\| \gamma_{\alpha\beta} \|$ $(\alpha, \beta = 1, 2)$ by δ, we get

$$\tau(L) = \frac{(8\pi G)^{1/2}}{c^2} \iint_F \sqrt{\rho\delta}\, dx^1 dx^2. \tag{6}$$

Since det $\| g_{ik} \| < 0$, the area bounded by the timelike cycle L cannot be arbitrarily small in the sense of this induced metric. We assume that the given cycle L has the minimal area and suppose that the density ρ changes little in the region F. We then get from (6) that

$$\tau(L) \approx \frac{(8\pi G\rho)^{1/2}}{c^2} \sigma(F),$$

where $\sigma(F)$ denotes the area of the region F. We now suppose that for the spatial length $l(L)$ of the cycle L and the area $\sigma(F)$ we have

$$\sigma(F) \sim \pi^{-1}[l(L)]^2. \tag{7}$$

It is then possible to assume that the components of the metric tensor for the surface F either depend only on $x^{1^2} + x^{2^2}$, or that the deviations from this condition have little effect on (7). We thus obtain the required relationship

$$\tau \sim 2 \cdot 10^{-24} \sqrt{\rho}\, l^2.$$

It thus follows that when $\rho \sim 10^{-31}$ g/cm^3, with $\tau \approx 1$ year, we have $l \sim$ [distance from the sun to the center of the galaxy] $\approx$ 8000 parsec; if however $l = 1000$ km, then $\tau \sim 6 \cdot 10^{-23}$ sec! If we take $\tau = 1$ year and $l = 1000$ km, we get $\rho \sim 10^{28}$ g/cm^3!! If we do not insist on condition (7), then with $\tau = 1$ year, $l = 1000$ km and $\rho \sim 10^{-31}$ g/cm^3, we get $\sigma \sim 10^9 \pi^{-1} l^2$. This means that the deviations from Euclidean geometry in a 3-space where timelike cycles occur are indeed vast and there can be little doubt about the conclusion that the situations in which timelike cycles occur lie outside the limits of contemporary knowledge.

4. We can give an example of a manifold with Euclidean topology and metric which admits timelike cycles.

Let

$$ds^2 = \frac{1}{2} dx^{0^2} + 2\Omega(x^2 dx^1 - x^1 dx^2)\, dx^0 + \left(\Omega^2 x^{2^2} - \frac{1}{2}\right) dx^{1^2}$$
$$- 2\Omega^2 x^1 x^2 dx^1 dx^2 + \left(\Omega^2 x^{1^2} - \frac{1}{2}\right) dx^{2^2} + \alpha\, dx^{3^2}.$$

The required curve is given by the relations $\{x^0, x^3 = \text{const}, x^1 = a \sin t, x^2 = a \cos t\}$, where $a \geq 1/\sqrt{2}\,\Omega$.

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