Докл. Акад. Наук СССР
Том 284 (1985), № 5

Soviet Math. Dokl.
Vol. 32 (1985), No. 2

# GROUPS OF ORDER AUTOMORPHISMS OF AFFINE SPACE AND THEIR DISCONTINUOUS EXTENSIONS

UDC 513.82

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We consider an $n$-dimensional affine space $A^n$, $n \geq 2$, in which there is specified a partial preorder $P$, invariant under all parallel translations; that is, a family of sets $P = \{P_x : x \in A^n\}$ satisfying the following conditions: 1) $x \in P_x$; 2) if $y \in P_x$, then $P_y \subset P_x$; and 3) if $t$ is a translation, then $t(P_x) = P_{t(x)}$ for any $x \in A^n$.

A bijection $f: A^n \to A^n$ is called an *order automorphism*, or a $P$-automorphism, if $f(P_x) = P_{f(x)}$ for every point $x \in A^n$. We denote the group of all $P$-automorphisms by $\mathrm{Aut}(P)$.

We pose the problem of describing the group $\mathrm{Aut}(P)$. The foundation for such research was laid by some papers of A. D. Aleksandrov (see the survey [1]). In this note we calculate the group $\mathrm{Aut}(P)$ for a disconnected preorder satisfying some additional axioms, and we give a classification of homogeneous disconnected preorders. The results we obtain extend the research begun in [2] and [3]. We also show how it is possible to extend the group $\mathrm{Aut}(P)$ to discontinuous bijections.

**1.** We fix a point $e$ for the whole article, and write $P$ instead of $P_e$. If $M$ in some set in $A^n$ containing $e$, then $M_x$ denotes the set obtained from $M$ by means of a translation $t$ such that $t(e) = x$. We denote by $\operatorname{int} A, \overline{A}, \partial A$ and $\operatorname{con} A$ respectively the interior, closure, boundary and convex hull of the set $A$. Also, $L(x, y)$ denotes the ray with origin $x$ passing through $y$, where $y \neq x$.

We say that the preorder $P = \{P_x : x \in A^n\}$ is *connected* if $x \in \overline{P_x \setminus \{x\}}$. Otherwise the preorder $P$ is *disconnected*. The preorder $P$ is *closed* (*open*) if $P$ is closed (respectively, $P \setminus \{e\}$ is open).

DEFINITION 1. A connected preorder $P$ is said to be $K$-*ruled*, where $K$ is a convex cone with $e$, if for any $x \in P$ we have $K_x \subset P$. A disconnected preorder $P$ is $K$-*ruled* if $K_x \subset P$ for any $x \in P \setminus \{e\}$.

If the realtion $y \in P_x$ is written in the form $x \leq y$, then $\leq$ is a partial preorder in $A^n$.

DEFINITION 2. A *displacement* $d_{El}$ (or $d_{EL}$), where $E$ is a hyperplane and $l$ is a vector (respectively, $L$ is a ray) not parallel to $E$, is a homeomorphism of $A^n$ onto itself that satisfies the following conditions:

a) $d_{El}$ ($d_{EL}$) is a translation on every hyperplane $E_a$.

b) $d_{El}$ ($d_{EL}$) takes segments (rays) equal and parallel to $l$ ($L$) into segments (rays) of the same kind.

DEFINITION 3. A *quasicylinder* $Q(E, l)$ is a set $M \subset A^n$ that satisfies the following conditions:

a) There are hyperplanes $\ldots, E_{-1}, E_0, E_1, \ldots$ parallel to $E$, where $E_{i+1}$ is obtained from $E_i$ by a translation by the vector $l$,

$$M = \bigcup_i [M_i \cup (M \cap E_i)], \tag{1}$$

where every $M_i$ is a cylinder formed by open segments equal to $l$ (as vectors) with ends on $E_i$ and $E_{i+1}$ (we do not exclude the case where some or even all the $M_i$ are empty).

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C75, 83C40.

b) $M$ does not admit representations (1) with the same hyperplane $E$ and vector $l'$ parallel to $l$ but larger than $l$.

The definition of a quasicylinder $Q(E, L)$, where $L$ is a ray, is similar.

DEFINITION 4. A preorder $P$ is said to be *affine* (*continuously affine*) if $\mathrm{Aut}(P) \subset \mathrm{Aff}(A^n)$; that is, it consists of affine transformations (respectively, every continuous $P$-automorphism is an affine transformation).

We introduce the following *weak Einstein axiom*:

$\mathrm{AE_w}$. *For any $x, y \in A^n$, if $y \in P_x$, then $P_x \cap P_y^-$ is bounded, where $P_x^- = \{y: y \leq x\}$.*

THEOREM 1. *Let $P = \{P_x: x \in A^n\}$ be a preorder in $A^n$, $n \geq 2$, that satisfies the axiom $\mathrm{AE_w}$ and is $K$-ruled, where $\mathrm{int}\, K \neq \varnothing$. Then either $P$ is a continuously affine order, or it is quasicylindric. If $P$ is the quasicylinder $Q(E_1, l_1), \dots, Q(E_p, l_p)$, then any continuous $P$-automorphism $f$ has the form*

$$f = f_0 \circ d_1 \circ \cdots \circ d_p, \tag{2}$$

*where $f_0$ is an affine transformation, and $d_i$ is the displacement $d_{E_i l_i}$. Any displacements $d_i$ are admissible, and the various $d_i$ commute (we allow some of the $l_i$ to be rays $L_i$).*

COROLLARY 1. *If $P$ is an open or closed preorder in $A^n$, $n \geq 2$, satisfying the conditions of Theorem 1, then either $P$ is an affine order or it is quasicylindric. If $P$ is the quasicylinder $Q(E_1, l_1), \dots, Q(E_p, l_p)$, then any $P$-automorphism $f$ has the form (2).*

**2.** Let $P$ be a preorder in $A^n$, $n \geq 2$. Then the cone

$$\exp P = \overline{\bigcup_{x \in P} L(e, x)}$$

with vertex $e$ is said to be *exterior*.

DEFINITION 5. A preorder $P$ is said to be *maximally ruled* if it is $\mathrm{ext}\, P$-ruled. A connected maximally ruled preorder is closed and conical. However, a disconnected maximally ruled preorder can be very arbitrary.

THEOREM 2. *If $P$ is a maximally ruled disconnected preorder with interior points, then*

$$\exp P = \bigcap_{e \in Q_x} Qx, \qquad Q_x = P_x \backslash \{x\},$$

*and consequently* $\mathrm{Aut}(P) \subset \mathrm{Aut}(\mathrm{ext}\, P)$.

We introduce the following *strong Einstein axiom*:

$\mathrm{AE_s}$. *The exterior cone* $\mathrm{ext}\, P$ *does not contain straight lines*; *and the homogeneity axiom*:

AH. *The stabilizer* $\mathrm{Aut}(P)_e$ *of the point $e$ acts transitively on $\partial Q_e$.*

An order that satisfies AH is said to be *homogeneous*.

THEOREM 3. *Let $P$ be a disconnected nontrivial homogeneous $K$-ruled* ($\mathrm{int}\, K \neq \varnothing$) *order in $A^n$, $n \geq 2$, satisfying the axiom $\mathrm{AE_w}$. Then the following assertions are true*:

1) *$P$ is affine, and $\overline{P}$ is a maximally ruled affine order.*

2) *Axiom $\mathrm{AE_s}$ holds.*

3) $\mathrm{ext}\, P$ *is a convex cone that admits an affine group $G$ of* $\mathrm{ext}\, P$*-automorphisms, acting transitively on* $\mathrm{int}(\mathrm{ext}\, P)$.

4) *The set $Q_e$ is convex and is isolated from the cone* $\mathrm{ext}\, P$ *by hyperplanes that cut off a finite volume from* $\mathrm{ext}\, P$.

5) $\mathrm{Aut}(P)_e$ *is a unimodular subgroup of $G$.*

Thus, it is possible to classify homogeneous disconnected $K$-ruled orders that satisfy $\mathrm{AE_w}$. This reduces to a classification of homogeneous convex cones in $A^n$, $n \geq 2$, and the calculation of the group $G$. Then $\partial Q_e$ is an orbit of the group $\mathrm{Aut}(P)_e \subset G$. The group $G$ was calculated by Vinberg in [4], Chapter III, §2, Proposition 1, and in [5].

**3. Discontinuous extensions of the group** $\mathrm{Aut}(P)$. Let $\mathfrak{A}$ be an *ideal* of subsets of $A^n$; that is, the following two conditions hold: 1) if $A, B \in \mathfrak{A}$, then $A \cup B \in \mathfrak{A}$; 2) if $A \subset B$ and $B \in \mathfrak{A}$, then $A \in \mathfrak{A}$.

We shall say that $A$ is *$\mathfrak{A}$-equivalent* to $B$ if the symmetric difference $A \triangle B \in \mathfrak{A}$. This is an equivalence relation that splits the set of subsets of $A^n$ into disjoint classes of $\mathfrak{A}$-equivalent sets. We denote the class with representative $A$ by $[A]$. We shall write $[A] \prec [B]$ if and only if $B \backslash A \in \mathfrak{A}$.

DEFINITION 6. The family of classes $\Pi = \{\pi_x : x \in A^n\}$, where $\pi_x$ is the class of $\mathfrak{A}$-equivalent sets associated with the point $x$, *specifies an $\mathfrak{A}$-preorder on $A^n$* if $[\{y : \pi_x \prec \pi_y\}] = \pi_x$ for all $x \in A^n \backslash a_0$, where $A_0 \in \mathfrak{A}$ is a fixed set.

DEFINITION 7. A bijection $f : A^n \to A^n$ is called an *$\mathfrak{A}$-order automorphism* if $f(\mathfrak{A}) \subset \mathfrak{A}$, $f^{-1}(\mathfrak{A}) \subset \mathfrak{A}$ and

$$f(S_x) \triangle S_{f(x)} \in \mathfrak{A} \quad (\text{or } [f(S_x)] = [S_{f(x)}])$$

for $x \notin A_f$, where $A_f \in \mathfrak{A}$, $A_0 \subset A_f \cap f(A_f)$ and

$$S_x = \{y \in A^n \backslash A_0 : \pi_x \prec \pi_y\}.$$

The set $\mathrm{Aut}(\Pi, \mathfrak{A})$ of all $\mathfrak{A}$-order automorphisms forms a group with composition as the group operation and the inverse bijection as inverse element.

If $P = \{P_x : x \in A^n\}$ is a preorder in $A^n$, then it specifies an $\mathfrak{A}$-preorder $\Pi_p = \{[P_x] : x \in A^n \backslash A_0\}$. It is therefore natural to call the group $\mathrm{Aut}(\Pi_p, \mathfrak{A})$ a discontinuous *extension* of the group of order automorphisms $\mathrm{Aut}(P)$.

Let $\mathfrak{A}_L$ denote the ideal of sets having zero Lebesgue measure.

THEOREM 4. *Let $C = \{C_x : x \in A^n\}$ be an order in $A^n$, $n \geq 3$, such that in rectangular Cartesian coordinates $x_0, x_1, \ldots, x_{n-1}$*

$$C_x = \left\{ y \in A^n : (y_0 - x_0)^2 - \sum_{i=1}^{n-1} (y_i - x_i)^2 \geq 0 \, and \; y_0 \geq x_0 \right\}. \tag{3}$$

*If $f : A^n \to A^n$ is a bijection of class $W_n^1(A^n)$ satisfying the equality*

$$(y_0 - x_0)^2 - \sum_{i=1}^{n-1} (y_i - x_i)^2 = [f_0(y) - f_0(x)]^2 - \sum_{i=1}^{n-1} [f_i(y) - f_i(x)]^2$$

*for almost all $y \in C_x$, where $x \in A^n \backslash A_0$, $A_0 \in \mathfrak{A}_L$, then $f$ is an $\mathfrak{A}_L$-order automorphism that coincides almost everywhere with an inhomogeneous Lorentz transformation.*

*Physical interpretation of Theorem* 4. We show that with the extension $\mathrm{Aut}(\Pi_C, \mathfrak{A}_L)$ there is associated a generalized principle of relativity, which substantially enriches the range of phenomena that fall under the action of the (special) principle of relativity. Later, instead of affine space we consider the four-dimensional arithmetic space $\mathbf{R}^4$.

As we know, the principle of relativity asserts that the laws governing natural phenomena are independent of the state of motion of the frame of reference so long as this motion is inertial, that is, a rectilinear uniform mechanical displacement in space. However, we can ask: is it not possible to extend the principle of relativity to motions that are not mechanical? By such motions we understand for the time being some forms of motion of matter other than a simple mechanical displacement. In this case, we must think of the "moving" frame of reference as the original "rest" frame of reference in which are "included" some forces whose nature we do not yet specify. According to this scheme Einstein in his time attempted to extend the theory of relativity to a gravitational field by replacing the field itself by nonlinear transformations of the coordinates ([6], §1). The possibility of introducing physical fields into the theory, associating discontinuous transformations with them, remained unused. An inertial mechanical motion is a diffeomorphism in $\mathrm{Aut}(C)$ (we omit such maps in future). Therefore, Theorem 4 suggests the following way of extending the Poincaré group $\Lambda$.

DEFINITION 8. Let $\tilde{\Lambda}$ be a *group of bijections* $f: \mathbf{R}^4 \to \mathbf{R}^4$ of class $W^1_{1,\mathrm{loc}}(\mathbf{R}^4)$ such that almost everywhere on $\mathbf{R}^4$

$$\sum_{i,k=0}^{3} \eta_{ik} \frac{\partial f_i}{\partial x^n} \frac{\partial f_k}{\partial x^n} = \eta_{nm}, \tag{4}$$

where $\eta_{ik} = \mathrm{diag}\{1, -1, -1, -1\}$, and $\partial f_i / \partial x^n$ is a generalized derivative.

It is not difficult to see that $\Lambda \subset \tilde{\Lambda}$. In $\tilde{\Lambda}$, as well as Lorentz transformations there occur discontinuous bijections that differ from Lorentz transformations on a set of measure zero [7].

In this case the generalized principle of relativity means that the equations describing physical laws must be invariant with respect to the group $\tilde{\Lambda}$. Invariance of differential equations with respect to the group $\tilde{\Lambda}$ must be understood in the generalized sense; that is, the integral identities by means of which generalized boundary-value problems are usually stated in mathematical physics must be invariant. For example, in the case of the Klein-Gordon equation

$$\sum_{i,k=0}^{3} \eta_{ik} \frac{\partial^2 \psi}{\partial x^i \partial x^k} - \mu^2 \psi = 0$$

we consider the identity

$$\int_{\mathbf{R}^4} \left( \sum_{i,k=0}^{3} \eta_{ik} \frac{\partial \psi}{\partial x^i} \frac{\partial \varphi}{\partial x^k} + \mu^2 \psi \varphi \right) d^4x = 0,$$

where $\psi \in W^1_{2,\mathrm{loc}}(\mathbf{R}^4), \varphi \in \overset{\circ}{W}{}^1_2(\Omega), \Omega \subset \mathbf{R}^4$ is an arbitrary bounded domain, and the integral is understood in Lebesgue's sense.

What new information can be given by the introduction of "spoiled" Lorentz transformations? We consider a bijection $\tilde{f} \in \tilde{\Lambda} \backslash \Lambda$ which is equivalent to the Lorentz transformation $f$. We assume that $\tilde{f}$ differs from $f$ only on a two-dimensional plane $\sigma$ that passes through the time-like line $\lambda$. It is quite possible that $\tilde{f}(\lambda)$ lies in the plane $\tilde{f}(\sigma)$, but is a space-like line. Thus, if in the "rest" frame of reference we "include the field" corresponding to the bijection $\tilde{f}$, then a conversion of an ordinary sublight velocity particle (tardyon) into a superlight velocity particle (tachyon) is observed. Here there is nothing unexpected, since in strong external fields ordinary particles are capable of revealing tachyon properties (see the survey [8], §3). What is important is that the kinematic description of tachyon interactions is possible in the language of discontinuous extensions of groups of order automorphisms.

There is a surprising consequence of the generalized principle of relativity. If $\lambda$ is a line in a 3-plane $\Sigma$, then we can choose $f \in \tilde{\Lambda} \backslash \Lambda$ so that $f(\lambda)$ is a set that is dense in the 3-plane $f(\Sigma)$. Consequently, the 3-trajectory of a particle with world line $\lambda$ under the action of the "field" $f$ is scattered over some spatial 2-plane; that is, it ceases to exist as a classical 3-trajectory of the particle. The inverse process is possible—"collecting" or "birth" of a particle from the space under "inclusion of the field" $f^{-1} \in \tilde{\Lambda} \backslash \Lambda$.

Therefore, we should not reduce the generalized principle of relativity merely to the description of interactions involving tachyons. It rather reflects the manifestation of properties of moving matter that are qualitatively different from those that are inherent in simple mechanical displacements.

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Received 29/AUG/84

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Translated by E. J. F. PRIMROSE